Invent math (2014) 197:1–45
DOI 10.1007/s00222-013-0477-9

Monodromy of cyclic coverings of the projective line

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Received: 4 November 2012 / Accepted: 21 June 2013 / Published online: 9 July 2013

Abstract We show that the image of the pure braid group under the monodromy action on the homology of a cyclic covering of degree d of the projective line is an arithmetic group provided the number of ramification points is sufficiently large compared to the degree d and the ramification degrees are co-prime to d.

Mathematics Subject Classification Primary 22E40 · Secondary 20F36

Contents

1 Introduction 2
 1.1 Description of the proof 6
2 Algebraic groups 8
 2.1 An inductive step for integral unitary groups 8
 2.2 Groups generated by complex reflections 10
 2.3 Some results on algebraic groups 12
 2.4 Products 14
3 Action of the braid group on a free group 14
 3.1 The pure braid group 15
 3.2 Artin's theorem 15
 3.3 Action on certain subgroups and sub-quotients 16
 3.4 The Gassner representation 16

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3.5 An invariant element in the Gassner representation 22
3.6 A supplement to the space of invariants 22
4 Properties of the Gassner representation 23
4.1 A (skew) Hermitian form preserved by the pure braid group . . 24
4.2 Irreducibility 27
5 Proof of Theorem 16 29
5.1 The reduced Gassner representation at roots of unity 29
5.2 Gassner representations with degenerate Hermitian forms 30
5.3 Proof of Theorem 16 34
6 Homology of cyclic coverings 36
6.1 Image of homology 37
6.2 Image of the Gassner module 39
7 Connection with monodromy 42
7.1 Some cyclic coverings of $\mathbb{P}^1$ 42
7.2 The compactification of $X_{a,k}$ 44
Acknowledgements 44
References 44

1 Introduction

A subgroup $\Gamma \subset GL_N(\mathbb{Z})$, is said to be an *arithmetic group* if Γ has finite index in its integral Zariski closure $\mathcal{G}(\mathbb{Z})$ (i.e. suppose $\mathcal{G} \subset GL_N$ is the Zariski closure of Γ; then $\Gamma \subset \mathcal{G}(\mathbb{Z})$, which by definition, is $\mathcal{G} \cap GL_N(\mathbb{Z})$. We say that Γ is arithmetic, if Γ is a subgroup of finite index in $\mathcal{G}(\mathbb{Z})$). Otherwise, we say that Γ is not arithmetic, or that Γ is *thin* [17].

A natural class of subgroups of $SL_N(\mathbb{Z})$ arise as monodromy groups. Suppose $X \to S$ is a family of smooth projective varieties $(X_s)_{s\in S}$ parametrised by a base variety S. Then the fundamental group $\pi_1(S)$ acts on the integral cohomology $H^*(X_s)$ of a typical fibre X_s. The image of this action is the "monodromy group". Griffiths and Schmid [7] first raised the possibility that monodromy groups are arithmetic. However, there are several examples which show that the monodromy group is not always an arithmetic group. Notable among them are those of Deligne-Mostow [6] (see also [15] where the monodromy group is not even finitely presented). In the examples of [6], the monodromy group is a subgroup of infinite index in an arithmetic lattice in a product of unitary groups $U(r, s)$ (such that the group of the real points of the Zariski closure of the monodromy group contains the product of the special unitary groups $SU(p, q)$ and) such that one of the factors of the product is $U(n-1, 1)$. The projection of the monodromy to this factor sometimes gives a lattice in $U(n-1, 1)$ which can be shown to be a non-arithmetic lattice in $U(n-1, 1)$.

The examples of [6] arise as the monodromy of certain families of cyclic coverings of a fixed order d of the projective line $\mathbb{P}^1(\mathbb{C})$, where the family is prescribed by choosing $n+1$ distinct branch points in the affine line $\mathbb{C}$, with fixed ramification. To be precise, let $n \geq 1$ and $d \geq 2$ be integers. Fix integers $k_1, k_2, \ldots, k_{n+1}$ with $1 \leq k_i \leq d-1$, and such that the g.c.d. of $k_1, k_2, \ldots, k_{n+1}$ and d is 1. Given $n+1$ distinct points $a_1, a_2, \ldots, a_{n+1}$ in the complex plane, put $a = (a_1, \ldots, a_{n+1}) \in \mathbb{C}^{n+1}$. Consider the curve $X_{a,k}$ given by the pair (x, y) satisfying the equation

$$y^d = (x-a_1)^{k_1}(x-a_2)^{k_2} \cdots (x-a_{n+1})^{k_{n+1}} \tag{1}$$

with $y \neq 0$ and $x \neq a_1, a_2, \ldots, a_{n+1}$.

Let $\mathcal{C}$ be the space of points in $\mathbb{C}^{n+1}$ all of whose coordinates are distinct; as the point $a \in \mathcal{C}$ varies, we get a family

$$\mathcal{F} = \left\{ (y, x, a) \in \mathbb{C}^* \times \mathbb{C} \times \mathcal{C} : y^d = \prod_{i=1}^{n+1} (x-a_i)^{k_i} \right\},$$

and the fibration $\mathcal{F} \to \mathcal{C}$ given by the projection map $(y, x, a) \mapsto a$. The fibre over a point $a \in \mathcal{C}$ is the affine curve given in (1). The curve $X_{a,k}$ is a compact Riemann surface $X^*_{a,k}$ minus a finite set of punctures. We may consider, analogously, the family $\mathcal{F}^*$ of compact Riemann surfaces $X^*_{a,k}$ fibering over $\mathcal{C}$.

The fundamental group of the space $\mathcal{C}$ is well known to be the pure braid group P_{n+1} on $n+1$ strands (see Sect. 7.1); thus the fibration $\mathcal{F}^* \to \mathcal{C}$ yields a monodromy representation

$$\rho^*_M(k, d) : P_{n+1} \to GL\big(H_1\big(X^*_{a,k}, \mathbb{Z}\big)\big),$$

of P_{n+1} on the integral homology of the fibre $X^*_{a,k}$. If N is the rank of the Abelian group $H_1(X^*_{a,k}, \mathbb{Z})$, then the image Γ of P_{n+1} is a subgroup of $GL_N(\mathbb{Z})$. It can be shown that the group $\mathcal{G}(\mathbb{R})$ of real points of the Zariski closure of Γ is contained in a product of unitary groups $U(p, q)$ such that $\mathcal{G}(\mathbb{R})$ contains the product of the special unitary groups $SU(p, q)$. The group $G = \mathbb{Z}/d\mathbb{Z}$ acts on the equation

$$y^d = \prod_{i=1}^{n+1} (x-a_i)^{k_i},$$

by the map $g(x, y) \mapsto gy$ for $g \in G$ where G is viewed as d-th roots of unity in $\mathbb{C}$. We may decompose the homology H_1 of $X^*_{a,k}$ into G eigenspaces with respect to this action. Fix a primitive d-th root of unity, say $\omega = e^{2\pi i/d}$. Fix a generator T of G. If $1 \le f \le d$ is an integer, fix the part of the homology $H_1(X^*_{a,k}, \mathbb{C})$ on which the generator $T \in G$ acts by the scalar ω^f. The group of real points of the Zariski closure of Γ acting on this part will again be contained in a unitary group of the form $U(p_f, q_f)$ and will (in general) contain the special unitary group $SU(p_f, q_f)$.

We now describe briefly, the results of [6]. Suppose f is an integer with $1 \le f \le d-1$, and coprime to d. Given $x \in \mathbb{R}$, denote by $\{x\}$ its fractional part. Put $\mu_i = \{\frac{k_i f}{d}\}$ for $1 \le i \le n+1$. Write $\mu_\infty = 2 - \sum \mu_i$. We impose the following conditions on the μ_i: (1) $\mu_i + \mu_j < 1$ for all i, j including $i = \infty$, (2) $0 < \mu_\infty$, (3) $\frac{1}{1-\mu_i-\mu_j}$ is an integer if $k_i \ne k_j$, and (4) if $k_i = k_j$ then $\frac{1}{1-2\mu_i}$ is a half integer.

Then it is shown in [6] that the factor of the group of real points of the Zariski closure of the monodromy Γ in $GL(H_1(X^*_{a,k}, \mathbb{Z})) = GL_N(\mathbb{Z}) \subset GL_N(\mathbb{C})$ corresponding to f (as in the preceding paragraph) contains the special unitary group $SU(n-1,1)$ and is contained in $U(n-1,1)$. Moreover, the projection of Γ to this factor gives a lattice in $U(n-1,1)$ (if the μ_i satisfy some further conditions, then the lattice in $U(n-1,1)$ is an *arithmetic* lattice).

For example, consider the family for varying $b_1, b_2, b_3, b_4 \in \mathbb{C}$, all distinct, of the curves corresponding to the equation

$$y^{18} = (x-b_1)(x-b_2)(x-b_3)(x-b_4).$$

In this case, $n = 3$. By the criteria of [6], the monodromy is non-arithmetic. In the notation of the preceding paragraph, we take $f = 7$ and $d = 18$; then $\mu_i = 7/18$ and $\mu_\infty = 8/18$. Hence $\frac{1}{1-\mu_i-\mu_j}$ is a half integer—namely 9/2—if $i, j \le 4$ (and hence $\mu_i = \mu_j$); and $\frac{1}{1-\mu_i-\mu_j}$ is an integer—namely 6—if $i \le 4$ and $j = \infty$. By the half integrality (Σ-*INT*) criterion of Mostow (in [14], see p. 104, with $N = 5$ and $\mu_i = 7/18$, and $\mu_\infty = 8/18$) it follows that the projection of Γ to the factor corresponding to $f = 7$ is a discrete subgroup of $U(2,1)$ and is, in fact, a lattice in $U(2,1)$. One can easily check, from the list given there, that the monodromy is non-arithmetic i.e. has infinite index in its integral Zariski closure.

Let us now return to the general situation of (1). The condition of [6] that $0 < \mu_\infty = 2 - \sum\{\frac{k_i f}{d}\}$ implies that $n+1 \le \sum k_i \le 2d$ and hence that $n \le 2d-1$. We would like to investigate what happens when $n \ge 2d$. The following theorem says that if $n \ge 2d$ then for most k_i's, the monodromy is arithmetic. Precisely, we prove

Theorem 1 *Suppose $d \ge 2$ and $n \ge 1$ are integers, $k_1, \dots, k_{n+1}$ are integers with $1 \le k_i \le d-1$ with $gcd(k_i, d) = 1$ for each i. Suppose that*

$$n \ge 2d.$$

Then the image $\Gamma = \rho^*_M(k,d)(P_{n+1})$ *of the monodromy representation* $\rho^*_M(k,d)$ *of* P_{n+1} *is an arithmetic group.*

Moreover, the monodromy group is (up to finite index) a product of irreducible lattices each of which has $\mathbb{Q}$*-rank at least two.*

In [21], the case when all the integers k_i are 1 was considered (then $gcd(k_i,d)=1$ for all i). Consider the compactification X^*_a of the affine curve

$$y^d = (x-a_1)(x-a_2)\cdots(x-a_{n+1}),$$

with $y \neq 0$ and $x \neq a_1,\ldots,a_{n+1}$. There is now the monodromy action of the pure braid group P_{n+1} (even of the full braid group B_{n+1}) on $H_1(X^*_a,\mathbb{Z})$. The following result is proved in [21].

Theorem 2 *If* $d \geq 3$ *and* $n \geq 2d$, *then the image* Γ *of the monodromy representation* $\rho(d): B_{n+1} \to GL(H_1(X^*_a,\mathbb{Z})) = GL_N(\mathbb{Z})$ *is an arithmetic group.*

Moreover, the monodromy is a finite index subgroup of a product of irreducible lattices, each of which is a non-co-compact arithmetic group and has $\mathbb{Q}$*-rank at least two.*

Remark 1 If $n+1 \leq d$ then the group of integral points of the Zariski closure of the monodromy is (up to finite index) a product of irreducible arithmetic lattices, some of which form *co-compact* lattices of their real Zariski closures.

A result of A'Campo [3] says that Theorem 2 holds when $d=2$ as well.

If we replace the pure braid group by the mapping class group Γ_g of the fundamental group of a compact Riemann surface of genus $g \geq 2$, and consider analogously, the action of Γ_g on the family of cyclic coverings of a fixed degree of the family of genus g Riemann surfaces, then the arithmeticity of the image of this action (monodromy) is proved in [12]. (At the time the present article was written, the author was not aware of the paper [12]; the method of proof is similar and uses the presence of unipotent elements in the monodromy group. But, in the present article, more work is needed to generate unipotent elements—under the assumption that $n \geq 2d$.)

A special case of Theorem 1 is the following

Corollary 1 *Suppose* d *is a prime,* $k_1,\ldots,k_{n+1}$ *integers with* $1 \leq k_i \leq d-1$ *and* $n \geq 2d$. *Then the monodromy group* Γ, *namely the image of* P_{n+1} *under the representation* $\rho(k,d): P_{n+1} \to GL(H_1(X^*_{a,k},\mathbb{Z}))$ *is an arithmetic group.*

Remark 2 If d is not assumed to be prime, then the analogue of Corollary 1 is false in general, even when n is large. As an example, consider $d = 2 \times 18$ and let n be *arbitrary*. Suppose $a_1, a_2, \ldots, a_n, b_1, b_2, b_3, b_4$ are distinct complex numbers. Consider the two equations

$$C_d(a,b):\quad y^{2\times 18} = \left(\prod_{i=1}^{n}(x-a_i)\right)^{18}(x-b_1)\cdots(x-b_4)\quad \text{and}$$

$$C_{18}(b):\qquad w^{18} = (x-b_1)\cdots(x-b_4).$$

There is a map $C_{2\times 18}(a,b) \to C_{18}(b)$ given by $(x,y)\mapsto(x,w)$ with

$$w = \frac{y^2}{(x-a_1)\cdots(x-a_n)}.$$

The monodromy of the family $C_{2\times 18}(a,b)$ (as a and b vary) on the first homology of the curves $C_{2\times 18}(a,b)$ maps onto the corresponding monodromy of the family of the curves $C_{18}(b)$ (as b varies). The latter is not arithmetic, by the example discussed earlier. Therefore, the monodromy of the family $C_{2\times 18}(a,b)$ is also non-arithmetic.

1.1 Description of the proof

The proof is very similar to the proof of Theorem 2 given in [21]. In [21] the proof was by showing that the monodromy was related to the Burau representation. The properties of the Burau representation (especially those at roots of unity) were used in the course of the proof.

Analogously, in the present paper, Theorem 1 is deduced from the arithmeticity of the images of certain representations (the reduced Gassner representation specialised at roots of unity defined in Sect. 3.4) of the pure braid group P_{n+1}. We also have to establish, somewhat precisely, the exact relationship of the monodromy in Theorem 1 with the Gassner representation. This is much more complicated than in the Burau case. The monodromy representation of Theorem 1 is related to the reduced Gassner representation of Theorem 16 as follows (see [9] for related results).

One can define the reduced Gassner representation $g_n(k,d)$ at d-th roots of unity where k is the $n+1$-tuple $(k_1,\ldots,k_{n+1})$. The image of $g_n(k,d)$ takes the pure braid group P_{n+1} into $GL_n(E_d)$ where $E_d=\mathbb{Q}(\omega_d)$ is the d-th cyclotomic extension. We will see in Sect. 4 that the Gassner representation $g_n(k,d)$ is irreducible if $\sum k_i$ is *not* divisible by d; if $\sum_i k_i$ is divisible by d, then $g_n(k,d)$ contains the one dimensional trivial representation $E_d v$ and the quotient, denoted $\overline{g}_n(k,d)$, is irreducible. By an abuse of notation, we write $\overline{g}_n(k,d)$ for the representation $g_n(k,d)$ even when $\sum k_i$ is not divisible by d.

If $X_{a,k}$ is the open curve, then we have the monodromy action, denoted $\rho_M(k,d)$ on $H_1(X_{a,k},\mathbb{Q})$ (and the action $\rho_M^*(X_{a,k}^*,\mathbb{Q})$ on the homology of the projective curve $X^*_{a,k}$). On the homology of $X_{a,k}$ the cyclic group $\mathbb{Z}/d\mathbb{Z}$ operates. Given a module V of the $\mathbb{Q}$-group algebra $\mathbb{Q}[\mathbb{Z}/d\mathbb{Z}]$, denote by V^{ni} the quotient of V modulo the space of invariant vectors in V under the action of $\mathbb{Z}/d\mathbb{Z}$. Take $V = H_1(X_{a,k}, \mathbb{Q})$. We call V^{ni} the "non-invariant" part of V and denote by $\rho_M(k,d)^{ni}$ the representation of P_{n+1} on V^{ni}. In Sect. 7, we will prove

Proposition 3 *Suppose that the numbers k_i are all co-prime to d. Denote by $\rho_M(k,d)$ the representation of the pure braid group P_{n+1} on the homology of the open curve $H_1(X_{a,k}, \mathbb{Q})$. Then the non-invariant part of $H_1(X_{a,k}, \mathbb{Q})$ is the direct sum*

$$\rho_M(k,d)^{ni} = \bigoplus_{e|d} g_n(k,e).$$

The representation $\rho^*_M(k,d)$ of P_{n+1} on the homology $H_1(X^*_{a,k}, \mathbb{Q})$ of the compact Riemann surface $X^*_{a,k}$ is the direct sum

$$\rho^*_M(k,d) = \bigoplus_{e|d} \overline{g}_n(k,e).$$

The sum is over all divisors $e \geq 2$ of the integer d.

Theorem 1 follows from Proposition 3 and Theorem 16.

The main section of the paper is Sect. 4. In Sect. 4, we show that the image of the pure braid group P_{n+1} at a primitive d-th root of unity contains many unipotent elements. More precisely, the proof of Theorem 16 is by showing that for $n \geq 2d$ the image $\Gamma_n(q)$ contains an arithmetic subgroup of the unipotent radical of a parabolic $\mathbb{Q}$- subgroup. By using results of Bass-Milnor-Serre and Tits [4, 18], and their extensions to other groups [16, 19, 20] on unipotent generators for noncocompact arithmetic groups of $\mathbb{R}$-rank at least two, one can then show that such groups are arithmetic if $n \geq 2d$.

In the Burau case, this was proved in [21]. It was possible to obtain unipotent elements in the Burau case when $n+1$ was divisible by d, since in that case, the Burau representation at d-th roots of unity is *degenerate*. The unitary group $U(h)$ of the relevant Hermitian form is not reductive and we can get, in the image of the Burau representation, elements which lie in the unipotent radical of $U(h)$.

An analogous result in the Gassner case is proved in the present paper. We exploit the fact that (if $n \geq 2d$) then a subrepresentation of the restriction of the Gassner representation at roots of unity, to a suitable smaller pure braid group, becomes degenerate. One can then generate unipotent elements. The existence of such a suitable smaller pure braid group is ensured by a pigeonhole argument if $n \geq 2d$. This is worked out in Sect. 4.

In Sect. 6 we relate the Gassner representation to the pure braid action on certain finite index subgroups of the free group on $n+1$ generators. This relation is obtained by using a Theorem of Artin on the action of the (pure) braid group on the free group on $n+1$ generators. We then relate this action to the monodromy in Sect. 7.

2 Algebraic groups

The following theorem is an extension to all simple groups, and all opposing parabolic subgroups, of a result of Bass-Milnor-Serre and of Tits (the theorem of Bass-Milnor-Serre and Tits was proved for SL_n ($n \geq 3$) and Sp_{2g} ($g \geq 2$)), and where the parabolic subgroup was a minimal parabolic subgroup. We refer to Sect. 2 of [21] for a detailed description and definitions of the terms involved.

Theorem 4 *Suppose G is an absolutely almost simple linear algebraic group defined over a number field K, such that K-rank of G is ≥ 1 and $G(O_K)$ has higher real rank, i.e.*

$$\infty - rank(G) \stackrel{def}{=} \sum_{v|\infty} K_v - rank(G) \geq 2.$$

Suppose P is a parabolic K-subgroup of G with unipotent radical U and let P^- be a parabolic K-subgroup defined over K and opposed to P with unipotent radical U^-. Let $\Gamma \subset G(O_K)$ be a subgroup which intersects $U(O_K)$ in a finite index subgroup (and similarly with $U^-(O_K)$). Then Γ has finite index in $G(O_K)$.

2.1 An inductive step for integral unitary groups

In this subsection, we prove a result which will be used in the inductive proof of Theorem 1. The result says that a subgroup of the integral unitary group has finite index if it contains finite index subgroups of smaller integral unitary groups. In the following, we will assume that E is a totally imaginary quadratic extension of a totally real number field K. Let $x \mapsto \overline{x}$ denote the action of the non-trivial element of the Galois group of the quadratic extension E/K. Assume that V is a finite dimensional E-vector space and that $h: V \times V \to E$ a K bilinear form such that $h(\lambda v, \mu w) = \lambda\overline{\mu}h(v, w)$ for all $\lambda, \mu \in E$ and all $v, w \in V$. Assume that $h(w, v) = \overline{h}(v, w)$ for all $v, w \in V$. Then the unitary group $U(h)$ (resp. the special unitary group $SU(h)$) of elements of $GL(V)$ (resp. $SL(V)$) which preserve h is naturally an algebraic group (resp. an almost simple algebraic group) over the totally real number field K. Under suitable conditions, we will be able to apply Theorem 4 to $SU(h)$.

Moreover, if $K_v \simeq \mathbb{R}$ is an Archimedean completion of the (totally real) number field K, then the base change of $U(h)$ to K_v is the usual unitary group of the Hermitian form over $\mathbb{R}$. In particular, the special unitary group $SU(h)(K_v)$ is a co-compact subgroup of $U(h)(K_v)$. As a consequence, the group of integral points $U(h)(O_K)$ and $SU(h)(O_K)$ are commensurable. Therefore, arithmetic subgroups of $U(h)$ or of $SU(h)$, are the same up to commensurability.

We note that in our applications, the groups involved will be unitary groups of *skew Hermitian forms*; but these are naturally isomorphic to unitary groups of Hermitian forms, by changing the skew form by a multiple of an imaginary element. For this reason, we do not stress the nature of the form, whether it is Hermitian or skew Hermitian.

Notation With the preceding assumptions, let $V = (V, h)$ be a nondegenerate Hermitian space over E such that $E - rank(V, h) \geq 2$; since the special unitary group is a K-group, this hypothesis is equivalent to $K - rank(SU(h)) \geq 2$. Let W, W' be codimension one subspaces on which h is again non-degenerate. Suppose that $\Gamma \subset H_V(O_K)$ is a subgroup such that its intersection with $U_W(O_K)$ (resp. $U_{W'}(O_K)$) has finite index in $U_W(O_K)$ (resp. in $U_{W'}(O_K)$).

Lemma 5 *With the preceding notation, suppose that there exists a non-degenerate subspace W'' of the intersection $W \cap W'$ which contains a nonzero isotropic vector v. Then the group Γ has finite index in $U_V(O_K)$.*

Proof By the non-degeneracy of h on $W'' \subset W \cap W'$ the space W'' contains a vector v^*, also isotropic, such that $h(v, v^*) = 1$. Write the orthogonal decomposition $V = (Ev + Ev^*) \oplus X$. Then $W = (Ev + Ev^*) \oplus X \cap W$ and similarly for W'.

Consider the filtration

$$0 \subset Ev \subset E \oplus X \subset Ev \oplus X \oplus Ev^* = V.$$

Denote the corresponding Heisenberg group (the unipotent subgroup of $U(V)$ which preserves the flag and acts by identity on successive quotients), by $H(V)$ and its integral points by $H(V)(O_K) = H_V(O_K)$. The group $P \subset U(V)$ which preserves the above partial flag is a parabolic subgroup and $H(V)$ is its unipotent radical. Define similarly, the smaller Heisenberg groups $H(W)$ and $H(W')$ and their integral points $H(W)(O_K)$ and $H(W')(O_K)$.

By assumption, $H(W) \cap \Gamma$ has finite index in $H(W)(O_K)$; similarly for $H(W')$. The two integral Heisenberg groups generate $H(W)(O_K)$ up to finite index, since two distinct vector subspaces of codimension one, span the whole space. We thus find that Γ contains a subgroup of finite index in the integral unipotent radical of a parabolic K subgroup.

Similarly, we find a finite index subgroup of an opposite integral unipotent radical in the group Γ. Therefore, by Theorem 4 applied to $SU(h)$, $\Gamma \cap SU(h)$ is arithmetic. Since $U(h)(O_K)$ and $SU(h)(O_K)$ are commensurable, it follows that Γ is an arithmetic subgroup of $U(h)(O_K)$. □

2.2 Groups generated by complex reflections

The results in this subsection deal with irreducibility of the action of groups generated by complex reflections on a complex vector space (sometimes ones equipped with a Hermitian form). They are essentially well known (cf. [13]), but we need a version involving additive subgroups of vector groups stable under complex reflections and therefore we record them here.

Let V be an n-dimensional vector space over a field K. We say that an element $T \in GL(V)$ is a *generalised reflection* if the endomorphism $T - 1$ has one dimensional image. Suppose that $T_1, \dots, T_n$ are generalised reflections such that $(T_i - 1)(V) = K\varepsilon_i$ and $\{\varepsilon_i : 1 \leq i \leq n\}$ form a basis of V. Assume that for each $i \leq n-1$, $(T_i - 1)(\varepsilon_{i+1}) = b_i\varepsilon_i$ with $b_i \neq 0$. Assume also that if $i \geq 2$ then $(T_i - 1)(\varepsilon_{i-1}) = a_i\varepsilon_i$ with $a_i \neq 0$.

Lemma 6

(1) *Let V and T_i be as in the preceding and Δ the group generated by the transformations $\{T_i; 1 \leq i \leq n\}$. Denote by V^Δ the space of vectors in V invariant under Γ. Then the quotient V/V^Δ is an irreducible representation of Δ.*

(2) *If in addition, we assume that $T_i(\varepsilon_j) = \varepsilon_j$ for all i, j with $|i - j| \geq 2$, then the space V^Δ of invariant vectors has dimension at most one.*

Proof Suppose that $W \neq V$ is a Δ invariant subspace. If $\varepsilon_j \in W$ for some j, by the T_{j-1} invariance of W, the vector $(T_{j-1} - 1)(\varepsilon_j)$ lies in W. By assumption, $(T_{j-1} - 1)\varepsilon_j$ is a non-zero multiple of ε_{j-1}; therefore, W contains ε_{j-1}. Similarly, $(T_{j+1} - 1)\varepsilon_j \in W$ and is a non-zero multiple of ε_{j+1} if $j \leq n-1$. Therefore, if ε_j lies in W for some j, then $\varepsilon_1, \dots, \varepsilon_n$ lie in W and hence $W = V$, a contradiction.

Consequently, W does not contain ε_j for any j. Consider the image $(T_i - 1)W \subset W$. If the image is non-zero, then it consists of all multiples of ε_i and this is impossible by the preceding paragraph. Therefore $(T_i - 1)W = 0$, which means that T_i is identity on W for every i. In other words, W is contained in V^Δ. This proves part (1) of the lemma.

We will now prove part (2), assuming (as in part (2)) that $T_i(\varepsilon_j) = \varepsilon_j$ if $|i - j| \geq 2$. Suppose that $v \in V^\Delta$ is of the form $v = x_2\varepsilon_2 + \cdots + x_n\varepsilon_n$ (i.e. the coefficient x_1 of ε_1 is zero). Applying (T_1-1) to v we get $0=(T_1-1)v=x_2b_1\varepsilon_1$ whence $x_2=0$. Now applying T_2-1 to v, we get $0=(T_2-1)v=x_3b_2\varepsilon_2$. Therefore $x_3=0,\ldots$. An easy induction now establishes that all the x_i are zero. Hence the linear map $V^{\Delta}\to K$ given by $v=\sum_{i=1}^{n}x_i\varepsilon_i\mapsto x_1$ (the first coordinate function) is injective. Therefore the second part of the lemma follows. □

We now prove a version of Lemma 6 for additive subgroups of a vector group stable under the T_i. Let A be an integral domain and Ω a field of characteristic zero containing A; suppose there is an involution of the field Ω (field automorphism of order two) which stabilises A and V a finite dimensional Ω vector space of dimension n with a *non-degenerate* Hermitian form h with respect to this involution. Suppose that $\{T_i\in GL(V):1\le i\le n\}$ preserve this Hermitian form such that the space of vectors fixed under T_i is of codimension one; then the image of T_i-1 is spanned by the unique (up to scalar multiples) eigenvector for T_i with eigenvalue not 1, denote it ε_i.

We will assume that the $\{\varepsilon_i:1\le i\le n\}$ form a basis of V, and that for each i,

$$T_i(\varepsilon_{i+1})=a_i\varepsilon_{i+1}+b_i\varepsilon_i \quad \text{with } b_i\neq 0,$$
$$T_i(\varepsilon_{i-1})=c_i\varepsilon_i+d_i\varepsilon_{i-1} \quad \text{with } c_i\neq 0.$$

Under these assumptions we have the

Lemma 7 *Let $\Gamma\subset U(h)(A)\subset U(V)$ be a subgroup generated by these complex reflections T_i. Let W be an additive subgroup of the vector group V, such that W is stable under the operators T_i. Then there exists a scalar $\lambda\neq 0$ in the integral domain A such that*

$$\lambda\varepsilon_1,\ldots,\lambda\varepsilon_n\in W.$$

In particular, Γ acts irreducibly on the vector space V; the representation is in fact absolutely irreducible.

Proof Not all the images $(T_i-1)W$ can be zero; for that would mean that all the vectors w in W are point-wise fixed by all the T_i; since distinct eigenspaces of a unitary operator are orthogonal, this means that w is orthogonal to ε_i for each i; therefore, $w=0$ since h is nondegenerate.

The image of (T_i-1) consists of multiples of ε_i. Therefore, there exists an integer i such that W contains a multiple $\lambda_i\varepsilon_i$ for some $\lambda_i\neq 0$. Since W is stable under all the T_j, the equation

$$T_{i-1}(\varepsilon_i)=a_{i-1}\varepsilon_i+b_{i-1}\varepsilon_{i-1},$$

shows that a multiple, namely $b_{i-1}\lambda_i\varepsilon_{i-1} = \lambda_{i-1}\varepsilon_{i-1}$ lies in $W, \dots,$ multiples of $\varepsilon_1, \dots, \varepsilon_i$ lie in W. Similarly, the equation

$$T_{i-1}(\varepsilon_i) = c_{i+1}(\varepsilon_{i+1}) + d_{i+1}\varepsilon_i, \quad d_{i+1} \neq 0,$$

shows that a nonzero multiple of ε_{i+1} lies in W, ..., a multiple of ε_n lies in W. This proves the first part of the lemma.

The foregoing proof also shows the irreducibility for any field Ω with an involution containing A in its fixed points. Since the fixed field of Ω under the involution may be embedded in an algebraically closed field F, and over F the unitary group becomes $GL_n(F)$, it follows that the irreducibility is true in this case as well: the action of Γ is absolutely irreducible. □

We will now derive a corollary of Lemma 7 which will be used later in the proof of (part (3) of) Proposition 18. We will keep the notation preceding (and including) Lemma 7. Denote by $\mathbb{Z}[\Gamma](\varepsilon_i)$ the additive subgroup of A^n spanned by the Γ translates of the vector ε_i.

Corollary 2 *Fix* $1 \leq i \leq n$. *Let* H_i *denote a subgroup of the group* A^* *of units of the integral domain* A *such that for every* $h \in H_i$ *there exists an element* $\gamma \in \Gamma$ *such that* $h\varepsilon_i = \gamma(\varepsilon_i)$.

(1) *There exists a* $\lambda = \lambda_i \neq 0$ *in* A *such that for every* j, *we have* $H_i\lambda\varepsilon_j \subset \mathbb{Z}[\Gamma](\varepsilon_i)$.
(2) *Suppose for* each i, H_i *is as in* [1]. *Let* H *be the subgroup of* A^* *generated by* $H_1, \dots, H_n$ (*then* $H = H_1 \cdots H_n$ *is the product*). *Then there exists a* $\lambda \neq 0$ *such that for every* $h \in H$ *and every* j, *the element* $\lambda h \varepsilon_j$ *lies in the* Γ *module generated by* $\varepsilon_1, \dots, \varepsilon_n$.

Proof An easy induction shows that (1) implies (2). We now prove (1).

Fix i. By Lemma 7, there exists a nonzero $\lambda \in A$ such that $\lambda\varepsilon_j \in \mathbb{Z}[\Gamma](\varepsilon_i)$. Let $h \in H_i$; by assumption, there exists $\gamma \in \Gamma$ such that $\gamma\varepsilon_i = h\varepsilon_i$. Therefore, $\lambda(h\varepsilon_j) = h(\lambda\varepsilon_j) \in h(\mathbb{Z}[\Gamma](\varepsilon_i)) = \mathbb{Z}[\Gamma](h\varepsilon_i) \subset \mathbb{Z}[\Gamma]\gamma\varepsilon_i = \mathbb{Z}[\Gamma](\varepsilon_i)$. □

2.3 Some results on algebraic groups

Let $U \subset SL_n(\mathbb{C})$ be the unipotent algebraic group consisting of the set of matrices u of the form

$$u = \begin{pmatrix} 1 & x_2 & \dots & x_n \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

This is the subgroup which preserves the partial flag

$$0 \subset \mathbb{C}e_1 \subset \mathbb{C}^n,$$

and acts trivially on successive quotients.

Proposition 8 *Let $H \subset SL_n(\mathbb{C})$ be a reductive algebraic subgroup which contains the unipotent algebraic group U. Then $H = SL_n(\mathbb{C})$.*

Proof Denote by T the group of diagonals in SL_n. The Lie algebra of SL_n splits into eigenspaces for the action of T, and the eigenvectors are E_{ij} and $E_{ii} - E_{jj}$ where E_{ij} is, in the usual notation, the $n \times n$ matrix whose ij-th entry is 1 and all other entries are zero. Then the Lie algebra $\mathfrak{u}$ of U is spanned by E_{1i} with $1 < i$.

Let $\mathfrak{h}$ be the Lie algebra of H. Write the direct sum decomposition $sl_n(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{h}'$ as modules under the adjoint action of H (we use the assumption that H is reductive). Since U is unipotent, if $\mathfrak{h}'$ is non-zero, there exists an $X \in \mathfrak{h}'$, $X \neq 0$, which is fixed by U. This means that the linear transformation X commutes with U.

The centraliser $\mathfrak{z}$ of U in $sl_n(\mathbb{C})$ is stable under the action of the diagonals T, since U is T-stable. Hence $\mathfrak{z}$ splits into eigenspaces for T. Since $E_{1i} \in Lie(U)$, the equation $[E_{1i}, E_{ij}] = E_{1j} \neq 0$ shows that the centraliser $\mathfrak{z}$ cannot contain E_{ij} with $i \geq 2$. It is also clear that the lie algebra of T acts faithfully on $Lie(U)$ under the adjoint action; therefore,

$$\mathfrak{z} = \bigoplus_{j \geq 2} \mathbb{C}E_{1j} = Lie(U) \subset \mathfrak{h}.$$

Hence X must lie in $\mathfrak{h}$; this is impossible and therefore, $\mathfrak{h}' = 0$. $\square$

Let V be an n-dimensional vector space over $\mathbb{C}$ and W, W' be two distinct codimension one subspaces and suppose we are given a decomposition $V = W \oplus \mathbb{C}v$ and $V = W' \oplus \mathbb{C}v'$. Assume that v, v' are linearly independent over $\mathbb{C}$. We will view $SL(W)$ (resp. $SL(W')$) as the subgroup of elements of $SL(V)$ which stabilise the subspace W (resp. W') and fix the vector v (resp. v').

Lemma 9 *$SL(V)$ is generated by $SL(W)$ and $SL(W')$.*

Proof Put $X = W \cap W'$. Then by our assumptions, X has codimension one in both W and W'. Fix a vector $w \in W$ (resp. $w' \in W'$) which does not lie in X. We have the decomposition $W = X \oplus \mathbb{C}w$ and $W' = X \oplus \mathbb{C}w'$. Write $E = \mathbb{C}w \oplus \mathbb{C}w'$. Then $sl(V) = sl(X) \oplus (X \otimes E^*) \oplus (X^* \otimes E) \oplus sl(E) \oplus Y$.

Here Y is the space of trace zero endomorphisms of V which act by a scalar on X and by a scalar on E.

If h is the sub-algebra generated by $sl(W)$ and $sl(W')$, then h (in fact the subspace $sl(W) + sl(W')$) contains $E \otimes X$ and $X^* \otimes E$ as subspaces; the sub-algebra generated by these subspaces contains $sl(E)$ (it is easy to capture the remaining one dimensional space Y in the subalgebra h). Therefore $h = sl(V)$. □

2.4 Products

The following lemma is proved in [21] and will be used in deducing the arithmeticity of the monodromy in Theorem 1 from the arithmeticity of the images of the Gassner representation at roots of unity (Theorem 16). After it was obtained, we learnt that this was already proved (in roughly the same form) in [12] and in [8].

Suppose X is a finite indexing set and for each element $p \in X$, let K_p be a number field and G_p be an absolutely almost simple group defined over K_p with $\infty - rank(G_e) \geq 2$. We assume that if $e, f \in X$ are distinct elements of X, then either K_e and K_f are not isomorphic as number fields or G_e and G_f are not isomorphic as algebraic groups over $K_e \simeq K_f$ (both groups thought of as algebraic groups over the same field $K_e \simeq K_f$).

Lemma 10 *With the preceding assumptions, suppose $\Gamma \subset \prod_{e \in X} G_e(O_e)$ is a subgroup of a product of higher rank arithmetic groups. Assume that for each $e \in X$, the projection of Γ in $G_e(O_e)$ has finite index in $G_e(O_e)$. Then Γ has finite index in the product.*

3 Action of the braid group on a free group

In this section, we first recall an action of the braid group on a free group, defined by Artin (see Sect. 3.2). This gives an action of the pure braid group on the first integral homology of the commutator subgroup of the free group. This action of the pure braid group is closely related to the Gassner representation. To make this relation precise, we need to replace the free group with the free product F of the free group with its Abelianisation. There is an action of the braid group on the kernel of the natural homomorphism from F onto the Abelianisation of the free group. The resulting action of the pure braid group on the homology of *this* kernel is identified with the Gassner representation (see Sect. 3.4). We can then define the reduced Gassner representation and construct a convenient basis $\{\varepsilon_i : 1 \leq i \leq n\}$ for it; we will use this basis in the next section to specialise the reduced Gassner representation at roots of unity.

3.1 The pure braid group

The braid group B_{n+1} on $n+1$ strands is the free group on the generators $s_1, s_2, \dots, s_n$ modulo the relations

$$s_i s_j = s_j s_i \quad \big(|i-j| \geq 2\big) \quad \text{and} \quad s_i s_j s_i = s_j s_i s_j \quad \big(|i-j| = 1\big).$$

The symmetric group S_{n+1} on $n+1$ symbols is the free group on the generators $\sigma_1, \dots, \sigma_n$ modulo the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \big(|i-j| \geq 2\big), \qquad \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \big(|i-j| = 1\big),$$

and the additional relations $\sigma_i^2 = 1$.

There is a natural surjective homomorphism $B_{n+1} \to S_{n+1}$ of the braid group onto the symmetric group on $n+1$ letters given by $s_i \mapsto \sigma_i$. The kernel of this homomorphism is the "Pure Braid Group" P_{n+1} on $n+1$ strands. The elements s_i^2 of B_{n+1} lie in P_{n+1}. It can easily be shown that the conjugates of these elements s_i^2 under all the elements of B_{n+1} generate P_{n+1}. If $i < j$ denote by Π_{ij} the product in B_{n+1} given by $\Pi_{ij} = s_{i+1} s_{i+2} \cdots s_{j-1}$. For $r < s$, set $A_{rs} = \Pi_{rs}^{-1} s_r^2 \Pi_{rs}$. In particular, $A_{r,r+1} = s_r^2$. The pure braid group is in fact generated by the elements A_{rs}.

3.2 Artin's theorem

Let F_{n+1} be the free group on $n+1$ generators $x_1, \dots, x_{n+1}$. The braid group B_{n+1} acts ([5], p. 21, Corollary (1.8.3)) on the free group F_{n+1} as follows.

$$\begin{aligned} &s_i(x_j) = x_j \quad \text{if } j \neq i, i+1, \\ &s_i(x_i) = x_i x_{i+1} x_i^{-1} \quad \text{and} \quad s_i(x_{i+1}) = x_i. \end{aligned}$$

The following theorem of Artin is fundamental to the rest of the section.

Theorem 11 *The above formulae give an action of the braid group on F_{n+1}; moreover, the action is faithful.*

The action of B_{n+1} is such that on the Abelianisation $\mathbb{Z}^{n+1}$ of F_{n+1}, the action is by the symmetric group S_{n+1} and the kernel of the map $B_{n+1} \to S_{n+1}$ is the pure braid group P_{n+1}.

The action of the generators $A_{r,s}$ of the pure braid group P_{n+1} can be worked out (from these formulae for the action of s_i) ([5], p. 25, Corollary 1.8.3):

$$A_{r,s}(x_i) = x_i \quad (i < r \text{ or } i > s), \qquad A_{r,s}(x_r) = (x_r x_s) x_r (x_r x_s)^{-1},$$

$$A_{r,s}(x_s) = x_r x_s x_r^{-1}, \qquad A_{r,s}(x_i) = [x_r, x_s]x_i[x_r, x_s]^{-1} \quad (r < i < s).$$

In particular, each generator x_i of F_{n+1} goes into a conjugate of itself under the action of P_{n+1}.

3.3 Action on certain subgroups and sub-quotients

Suppose F is a group, and Q a quotient of F and K the kernel of the quotient map $F \to Q$. Then there is the exact sequence

$$1 \to K \to F \to Q \to 1.$$

Denote by $K^1 = [K, K]$ the commutator subgroup of K and by $K^{ab} = K/K^1$ the Abelianisation of K. Then the conjugation action of F stabilises $[K, K]$ and F acts on K^{ab}. We may write K^{ab} additively. The action of F on K^{ab} is such that K acts trivially; hence the action of F on K^{ab} descends to an action of Q on K^{ab} and hence K^{ab} becomes a $\mathbb{Z}[Q]$-module where $\mathbb{Z}[Q]$ is the group ring of Q with $\mathbb{Z}$-coefficients.

We have an exact sequence

$$0 \to K^{ab} \to F/K^1 \to Q \to 1.$$

Suppose $H \subset Aut(F)$ be a subgroup of the automorphism group of F such that H stabilises K and *acts trivially* on Q; then H acts on the foregoing exact sequence and the action of H on K^{ab} commutes with the action of Q on K^{ab}; therefore, H acts by $\mathbb{Z}[Q]$-module maps on the Q module K^{ab}.

3.4 The Gassner representation

In the notation of Sect. 3.3, we take $F = F_{n+1} * F_{n+1}^{ab}$ to be the free product of the group F_{n+1} and its Abelianisation F_{n+1}^{ab} (written multiplicatively). Write, temporarily, H for F_{n+1}. The Abelianisation of F is $H^{ab} \times H^{ab}$. There is the multiplication map $m : H^{ab} \times H^{ab}$ given by $(x, y) \mapsto xy$. We have the composite map $\phi : F = H * H^{ab} \to H^{ab} \times H^{ab} \xrightarrow{m} H^{ab}$. This is a surjection with kernel K, say. We then have a *split* exact sequence

$$1 \to K \to H * H^{ab} \to H^{ab} \to 1.$$

The group in the middle is then a semi-direct product $H * H^{ab} \simeq K \rtimes H^{ab}$, since the exact sequence splits. Write the elements of the semi-direct product as a pair (w, t) with $w \in K$ and $t \in H^{ab}$. Write the image of the standard generators x_i of H in this semi-direct product group as a pair $x_i = (y_i, X_i)$. Therefore y_i and X_j generate the group F and hence the y_i generate K as a normal subgroup of F.

As in Sect. 3.3, we take the quotient of the group F by the commutator subgroup $[K, K]$, and get an exact sequence

$$0 \to K^{ab} \to \frac{H * H^{ab}}{[K, K]} \to H^{ab} \to 1,$$

which is still split over H^{ab}. Hence we may write the group in the middle as a semi-direct product $F/[K, K] = K^{ab} \rtimes H^{ab}$. An element of this group is written as a pair (w, t) with $w \in K^{ab}$ and $t \in H^{ab}$; the conjugation by t on K^{ab} is simply multiplying by the element t, when we view K^{ab} as a module over the group ring $\mathbb{Z}[H^{ab}]$. We write $H^{ab} = F_{n+1}^{ab}$ multiplicatively in the form $H^{ab} = X_1^{\mathbb{Z}} X_2^{\mathbb{Z}} \cdots X_{n+1}^{\mathbb{Z}}$. Denote by e_i the image of $y_i \in K$ in the Abelianisation K^{ab}. By the conclusion of the last paragraph, the elements e_i generate K^{ab} as a module over the group ring $\mathbb{Z}[H^{ab}]$:

$$K^{ab} = \sum_{i=1}^{n+1} \mathbb{Z}\big[H^{ab}\big](e_i).$$

We will now show that K^{ab} is a free module over R with e_i as basis. Write $R = \mathbb{Z}[H^{ab}]$. We will view R as a module over the multiplicative group H^{ab} by the formula $x(f_1, \ldots, f_{n+1}) = (xf_1, \ldots, xf_{n+1})$ where $x \in H^{ab}$ is viewed as a unit in R. Let $(\xi_i)_{i \le n+1}$ be the standard basis of R^{n+1}. Form the semi-direct product $\mathcal{H} = R^{n+1} \rtimes H^{ab}$. We then get a homomorphism from the free product $H * H^{ab}$ into $\mathcal{H}$ by specifying the homomorphism on the generators $x_i \mapsto (\xi_i, X_i)$ and $t \mapsto (0, t) \in \mathcal{H} = R^{n+1} \rtimes F_{n+1}^{ab}$. Then we get a homomorphism $H * H^{ab}$ which takes y_i to the element ξ_i. Therefore, we get a homomorphism of R modules from K^{ab} into R^{n+1} which sends e_i into the basis element ξ_i. This shows that the e_i are linearly independent over R; the last line of the preceding paragraph tells us that the e_i span K^{ab}. Hence the e_i form a basis of K^{ab} and

$$K^{ab} = R^{n+1} = \bigoplus_{i=1}^{n+1} Re_i.$$

We now write the product of two elements $x = (w, t)$, $y = (w', t') \in \overline{F}_{n+1} = K^{ab} \rtimes H^{ab}$. The product is given by

$$xy = (w, t)\big(w', t'\big) = \big(w + tw't^{-1}, tt'\big) = \big(w + t\big(w'\big), tt'\big).$$

The inverse of $x = (w, t)$ is $x^{-1} = (-t^{-1}w, t^{-1})$. An easy induction shows that

$$(v_1, t_1)(v_2, t_2) \cdots (v_{n+1}, t_{n+1}) = \left(\sum_{i=1}^{n+1} t_1 t_2 \cdots t_{i-1} v_i, t_1 t_2 \cdots t_{n+1} \right).$$

In this formula, v_i are vectors in K^{ab} and K^{ab} is viewed as an R-module. The following lemma is an immediate consequence of these formulae and the formulae in Sect. 3.2.

Lemma 12 *Let $x_r, x_s \in F_{n+1} = H$ be as before and $(e_r, X_r) = \overline{x}_r$, $(e_s, X_s) = \overline{x}_s \in \overline{F} = F/[K, K]$ be their images in the quotient group $F/[K, K]$ (which is a semi-direct product). Then we have the formulae (read in $\overline{F}$)*

$$\overline{x_r x_s x_r^{-1}} = (e_r + X_r e_s - X_s e_r, X_s) = \big((1 - X_s)e_r + X_r e_s, X_s\big), \tag{2}$$

$$x_r x_s x_r x_s^{-1} x_r^{-1} = \big(e_r + X_r(e_s) + X_r X_s(e_r) - X_r^2(e_s) - X_r e_r, X_r\big), \tag{3}$$

$$\overline{[x_r, x_s]} = (1 - X_s)e_r - (1 - X_r)e_s \quad \textit{and} \tag{4}$$

$$\overline{[x_r, x_s]x_i[x_r, x_s]^{-1}} = \big(e_i + (1 - X_i)v_{r,s}, X_i\big) \tag{5}$$

$$= \big(e_i + (1 - X_i)(1 - X_s)e_r - (1 - X_i)(1 - X_r)e_s, X_i\big). \tag{6}$$

The braid group B_{n+1} acts on the free group F_{n+1} and hence acts naturally on the free product $F = F_{n+1} * F_{n+1}^{ab}$. The preceding map $F \to F^{ab} = H^{ab} \times H^{ab} \to H^{ab}$ is equivariant for the action of B_{n+1} (and B_{n+1} acts via the finite group S_{n+1} on the Abelianisation F_{n+1}^{ab}). Hence B_{n+1} acts on the exact sequence

$$1 \to K \to F \to H^{ab} \to 1,$$

and the pure braid group P_{n+1} acts trivially on $F_{n+1}^{ab} = H^{ab}$. We are therefore in the situation of Sect. 3.3, and hence, as in Sect. 3.3, the group P_{n+1} acts by $\mathbb{Z}[F_{n+1}^{ab}] = R$-module maps on K^{ab}. We can now compute the action of the standard generators $A_{r,s}$ of the pure braid group P_{n+1} on the images of x_i in the quotient group $F/[K, K]$.

(1) Recall that $A_{r,s}(x_i) = x_i$ if $i \leq r - 1$ or $i \geq s + 1$. From Lemma 12, it follows that $A_{r,s}(e_i) = e_i$ for these i.

(2) $A_{r,s}(x_r) =^{x_r x_s} (x_r)$. When this equation is read modulo $[K, K]$, we see from Lemma 12 that

$$A_{r,s}(e_r)X_r = A_{r,s}(e_r X_r) = A_{r,s}(x_r) = e_r X_r e_s X_s (e_r X_r) X_s^{-1} e_s^{-1} X_r^{-1} e_r^{-1}.$$

Cancelling X_r on the right on the left most and right most sides of this equation, we see that

$$A_{r,s}(e_r) = (1 - X_r + X_r X_s)e_r + X_r(1 - X_r)e_s.$$

(3) The equation $A_{r,s}(x_s) =^{x_r} (x_s)$ becomes, modulo the subgroup $[K, K]$, the equation

$$A_{r,s}(e_s)X_s = \big((1 - X_s)e_r + X_r e_s\big)X_s.$$

(4) If $r < i < s$ then by Lemma 12,

$$\begin{aligned} A_{r,s}(e_i)X_i &= A_{r,s}(x_i) =^{[x_r, x_s]} (x_i) \\ &= \big(e_i + (1 - X_i)\big((1 - X_s)e_r - (1 - X_r)e_s\big)\big)X_i \end{aligned}$$

or

$$A_{r,s}(e_i) = e_i + (1 - X_i)\big((1 - X_s)e_r - (1 - X_r)e_s\big).$$

These equations imply that with respect to the basis e_i of $K^{ab} = R^{n+1}$, the action by P_{n+1} on the R module K^{ab} is exactly the *Gassner representation* $G_n(X) : P_{n+1} \to GL_{n+1}(R)$ (see [5], p. 119, formulae (3)–(24)).

Notation The ring $R = \mathbb{Z}[X_1^{\pm 1}, \ldots, X_{n+1}^{\pm 1}]$ of Laurent polynomials in $n + 1$ variables with integral coefficients, is an integral domain. Let $\Omega = \mathbb{Q}(X_1, \ldots, X_{n+1})$ be its field of fractions. We have the free R-module $K^{ab} = \sum_{i=1}^{n+1} Re_i$. This may be thought of as an R submodule of the Ω vector space $K^{ab} \otimes_R \Omega = \sum_{i=1}^{n+1} \Omega e_i$. We may write $e_i = (1 - X_i)v_i$ for some vector $v_i \in K^{ab} \otimes \Omega$.

Now the full braid group B_{n+1} acts on R via S_{n+1} by permuting the indices X_i of the generators (and the pure braid group acts trivially). Hence B_{n+1} acts on Ω by field automorphisms. We denote this action, for $g \in B_{n+1}$ and $\lambda \in \Omega$, by $(g, \lambda) \mapsto g(\lambda)$. The action of B_{n+1} on $K^{ab} \otimes \Omega$ is not linear over Ω but is "twisted linear": if $\lambda \in \Omega$, $g \in B_{n+1}$ and $w \in K^{ab}$, then $g(\lambda w) = g(\lambda)g(w)$.

Lemma 13 *Let $v_i = \frac{1}{1-X_i}e_i$ with e_i and R as before. The R module $\bigoplus_{i=1}^{n+1} Rv_i$ is stable under the action of P_{n+1}.*

Proof Since P_{n+1} acts by Ω-linear maps, it suffices to show that for every $g \in P_{n+1}$ and every v_i the translate $g(v_i)$ is an R linear combination of the v_j. We will in fact prove more; we will show that for every generator s_i of the *full braid group* B_{n+1}, the translate $s_i(v_j)$ is an R linear combination of the v_k. The group B_{n+1} acts by twisted Ω linear maps as before and not by Ω linear maps; however, it takes an element $\lambda w \in K^{ab}$, with $\lambda \in R$ and $w \in K^{ab}$ into an element of the form $\mu g(w)$ and hence preserves the space $\sum Rv_i$ provided each v_j is mapped into an R linear combination of the v_k. We now need only check that for each generator s_i of B_{n+1} and each v_j, the translate $s_i(v_j)$ is an R linear combination of the vectors $v_1, \dots, v_{n+1}$.

Suppose $j \neq i, i+1$. We have $s_i(x_j) = x_j$ for $j \neq i, i+1$. Therefore, $s_i(X_j) = X_j$. Since $x_i = (e_i, X_i)$ it follows that $s_i(e_j) = e_j$. We now write $e_j = (1 - X_j)v_j$ and note that s_i acts trivially on X_j. Hence

$$\begin{aligned}(1 - X_j)v_j = e_j = s_i(e_j) &= s_i\big((1 - X_j)v_j\big) \\ &= \big(1 - s_i(X_j)\big)s_i(v_j) = (1 - X_j)s_i(v_j).\end{aligned}$$

This shows that $s_i(v_j) = v_j$.

Suppose $j = i$. Then $s_i(x_i) = x_i x_{i+1} x_i^{-1} = [x_i, x_{i+1}]x_{i+1}$. We have expressed a commutator in terms of the e_j (see (4) in Lemma 12): hence the commutator $[x_i, x_{i+1}] = (1 - X_{i+1})e_i - (1 - X_i)e_{i+1}$. Therefore,

$$\big(s_i(e_i), X_{i+1}\big) = s_i\big((e_i, X_i)\big) = \big((1 - X_{i+1})e_i - (1 - X_i)e_{i+1} + e_{i+1}, X_{i+1}\big).$$

Comparing the extreme left and right hand sides of this equation, we see that $s_i(e_i) = (1 - X_{i+1})e_i + X_i e_{i+1}$. Now write $e_i = (1 - X_i)v_i$ and similarly for e_{i+1}. Then we have

$$\begin{aligned}(1 - X_{i+1})s_i(v_j) = s_i\big((1 - X_i)v_i\big) = s_i(e_i) &= (1 - X_{i+1})e_i + X_i e_{i+1} \\ &= (1 - X_{i+1})(1 - X_i)v_i + X_i(1 - X_{i+1})v_{i+1}.\end{aligned}$$

Cancelling $(1 - X_{i+1})$ on both the extreme right and left hand sides of this equation, we get

$$s_i(v_i) = (1 - X_i)v_i + X_i v_{i+1}.$$

Suppose $j = i + 1$. Then $s_i(x_{i+1}) = x_i$. Reading this as in Lemma 12 we get $s_i(e_{i+1}, X_{i+1}) = (e_i, X_i)$ and $s_i(X_{i+1}) = X_i$. Therefore, we get $s_i(e_{i+1}) = e_i$. Writing $e_i(1 - X_i)v_i$ we see that

$$(1 - X_i)s_i(v_{i+1}) = s_i\big((1 - X_{i+1})v_{i+1}\big) = s_i(e_{i+1}) = e_i = (1 - X_i)v_i.$$

Comparing the extreme right and left hand sides of this equation, we get

$$s_i(v_{i+1}) = v_i.$$

From the last three paragraphs, we see that each $s_i(v_j)$ is an R-linear combination of the v_j. Therefore, the lemma follows. $\square$

Lemma 14 *Set $\varepsilon_i = v_i - v_{i+1}$ for $1 \le i \le n$. Then the R module generated by $\{\varepsilon_i : 1 \le i \le n\}$ is stable under the action of P_{n+1}.*

In particular, with respect to the basis $\varepsilon_1, \dots, \varepsilon_n$ the transformation $T_i = s_i^2$ has the matrix form

$$\begin{pmatrix} 1 & 0 & 0 \\ X_i(1 - X_{i+1}) & X_i X_{i+1} & 1 - X_i \\ 0 & 0 & 1 \end{pmatrix} \oplus 1_{n-3},$$

where the 3×3 matrix is with respect to the basis elements $\varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}$ and s_i^2 acts as identity on the basis elements ε_j for the other indices j. In particular, s_i^2 are complex reflections.

Proof As in the proof of Lemma 13, because the full braid group acts by "twisted" Ω linear maps on $K^{ab} \otimes \Omega$, it suffices to check that the full braid group preserves the R module spanned by the ε_i.

We use the formulae in the proof of Lemma 13. Suppose $j + 1 < i$. Then

$$s_i(\varepsilon_j) = s_i(v_j - v_{j+1}) = v_j - v_{j+1} = \varepsilon_j.$$

Similarly, $s_i(\varepsilon_j) = \varepsilon_j$ if $j > i + 1$. We now get from the formulae for $s_i(v_i)$ and $s_i(v_{i+1})$ obtained from Lemma 13, that

$$s_i(\varepsilon_i) = s_i(v_i - v_{i+1}) = (1 - X_i)v_i + X_i v_{i+1} - v_i = -X_i \varepsilon_i,$$
$$s_i(\varepsilon_{i+1}) = s_i(v_{i+1} - v_{i+2}) = v_i - v_{i+2} = \varepsilon_i + \varepsilon_{i+1}.$$

Finally,

$$s_i(\varepsilon_{i-1}) = s_i(v_{i-1} - v_i) = v_{i-1} - (1 - X_i)v_i - X_i v_{i+1} = \varepsilon_{i-1} + X_i \varepsilon_i.$$

This proves the first part of the lemma.

To prove the second part, we must compute $s_i^2(\varepsilon_j)$. If $j + 1 < i$ or if $j > i + 1$ then $s_i(\varepsilon_j) = \varepsilon_j$; hence $s_i^2(\varepsilon_j) = \varepsilon_j$. We now compute $s_i^2(\varepsilon_{i-1})$:

$$\begin{aligned} s_i\big(s_i(\varepsilon_{i-1})\big) &= s_i(\varepsilon_{i-1} + X_i \varepsilon_i) = s_i(\varepsilon_{i-1}) + X_{i+1} s_i(\varepsilon_i) \\ &= \varepsilon_{i-1} + X_i \varepsilon_i + X_{i+1}(-X_i \varepsilon_i) = \varepsilon_{i-1} + X_i(1 - X_{i+1})\varepsilon_i. \end{aligned}$$

Next we compute

$$s_i^2(\varepsilon_i) = s_i(-X_i \varepsilon_i) = -X_{i+1} s_i(\varepsilon_i) = X_{i+1} X_i \varepsilon_i.$$

Finally,

$$s_i^2(\varepsilon_{i+1}) = s_i(\varepsilon_i + \varepsilon_{i+1}) = -X_i\varepsilon_i + \varepsilon_i + \varepsilon_{i+1} = (1 - X_i)\varepsilon_i + \varepsilon_{i+1}. \quad \square$$

3.5 An invariant element in the Gassner representation

The product element $x_1x_2\cdots x_{n+1} \in F_{n+1}$ is invariant under the action of the braid group B_{n+1}. The image of $x_1x_2\cdots x_{n+1}$ in the semi-direct product group $F/[K, K]$ is therefore invariant; since the image of x_i is written as (e_i, X_i), it follows from the formulae before Lemma 12 that

$$\begin{aligned} x_1x_2\cdots x_{n+1} &= (e_1X_1)(e_2X_2)\cdots(e_{n+1}X_{n+1}) \\ &= \big(e_1 + X_1(e_2) + X_1X_2(e_3) + \cdots + X_1X_2\cdots X_n(e_{n+1}), \\ &\quad X_1X_2\cdots X_{n+1}\big). \end{aligned}$$

Therefore, the element

$$v = \sum_{i=1}^{n+1} X_1X_2\cdots X_{i-1}(e_i) \in K^{ab} \tag{7}$$

is invariant under the action of the pure braid group. Moreover, since all the coefficients $X_1X_2\cdots X_i$ of v are units in the ring

$$R = \mathbb{Z}\big[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_{n+1}^{\pm 1}\big],$$

it follows that v is part of a basis of $K^{ab} = \bigoplus_{i=1}^{n+1} Re_i = R^{n+1}$. Consider the quotient module $V_n(X) = K^{ab}/Rv$. Then $V_n(X) \simeq R^n$ and is a module over P_{n+1}. Let Ω be the field of fractions of the integral domain R. Then $V_n(X) \otimes_R \Omega$ is called the *reduced Gassner Representation* over Ω the field of fractions, and is denoted

$$g_n(X) : P_{n+1} \to GL_n(\Omega).$$

3.6 A supplement to the space of invariants

We will now find a sub-module $W_n(X)$ which has zero intersection with the space Rv of multiples of the invariant vector v in the (non-reduced) Gassner representation, which is stable under the action of the pure braid group P_{n+1} and is free of rank n over R. We will view $K^{ab} = \bigoplus_{i=1}^{n+1} Re_i$ as a subgroup of the Ω vector space $\Omega^{n+1} = K^{ab} \otimes_R \Omega = \bigoplus_{i=1}^{n+1} \Omega e_i$. Write $e_i = (1 - X_i)v_i$, with $v \in \Omega^{n+1}$.

The calculations of Lemma 12 show that the R-module $L = \bigoplus_{i=1}^{n+1} Rv_i$ is stable under the action of the pure braid group P_{n+1}. The lemma also implies that the free R sub-module

$$W = \bigoplus_{i=1}^{n} R\varepsilon_i \simeq R^n,$$

(where $\varepsilon_i = v_i - v_{i+1}$) is stable under P_{n+1}. We note that the commutator $[x_i, x_{i+1}]$, (see 4 in Lemma 12) viewed as an element of the kernel K^{ab} has the form

$$\begin{aligned} [x_i, x_{i+1}] &= (1 - X_i)e_{i+1} - (1 - X_{i+1})e_i \\ &= (1 - X_i)(1 - X_{i+1})(v_i - v_{i+1}) \end{aligned} \tag{8}$$

i.e.

$$[x_i, x_{i+1}] = (1 - X_i)(1 - X_{i+1})\varepsilon_i. \tag{9}$$

As an R module, W is a summand of L: $L = W \oplus Rv_{n+1}$ (since $\varepsilon_i = v_i - v_{i+1}$ form a basis of W). It can be proved that W is *not* a direct summand as a module over P_{n+1}; however, it is so, when all the modules are tensored with the field Ω:

$$L \otimes \Omega = \Omega^{n+1} = W \otimes \Omega \oplus \Omega v,$$

where $v = \sum_{i=1}^{n+1} X_1 \cdots X_{i-1} e_i$. Write $\pi = X_1 \cdots X_i$ and $e_i = (1 - X_i)v_i$ as before. Then the invariant element v of 7 can be written as a linear combination of ε_i and e_{n+1}:

$$v = (1 - \pi_1)\varepsilon_1 + (1 - \pi_2)\varepsilon_2 + \cdots + (1 - \pi_n)\varepsilon_n + (1 - \pi_{n+1})v_{n+1} \in L. \tag{10}$$

We will refer to the P_{n+1} module $W = \bigoplus R\varepsilon_i \simeq R^n$ as the *reduced Gassner representation* over the ring R. We will later consider the reduction (modulo ideals of R) of W to obtain specialisations of the reduced Gassner representation. Over the fraction field Ω, the representation $W \otimes \Omega$ is isomorphic to the quotient $K^{ab} \otimes \Omega / \Omega v$ and hence the terminology is consistent with the end of the preceding subsection.

4 Properties of the Gassner representation

In this section, we prove some properties of the Gassner representation and its specialisations. We first *construct* (in Sect. 4.1) a skew Hermitian form on the reduced Gassner representation, which is invariant under the action of the pure braid group. The existence of the form is due to [10] (see also [6]), but the construction for the basis ε_i defined in the previous section, may perhaps be new. Using this form, it is easy to decide when specialisations of the Gassner representation (especially d-th roots of unity) are irreducible.

It turns out that the specialised reduced Gassner representation is irreducible if and only if the form is nondegenerate. When it is degenerate, we will get, in the image of the Gassner representation, many unipotent elements (Proposition 19). This is crucial to our proof of arithmeticity (Theorem 16).

4.1 A (skew) Hermitian form preserved by the pure braid group

It is known (see [10], Theorem 3.3) that the Gassner representation has a skew-Hermitian form invariant under the action of the pure braid group. To compute this form with respect to the "ε"-basis $\{\varepsilon_i : 1 \le i \le n\}$ of the reduced Gassner representation, we proceed as follows. Since the matrices s_i^2 act by complex reflections, and ε_i is the unique (up to scalar multiples) eigenvector with eigenvalue $X_i X_{i+1} \neq 1$, it follows from Lemma 7 that the group generated by the s_i^2 acts already irreducibly on $R^n \otimes \Omega$, where Ω is the field of fractions of R. Further, since s_i^2 are unitary, the eigenvectors with eigenvalue $= 1$ of s_i^2 are orthogonal with respect to h, to the eigenvector ε_i. Hence ε_j and ε_i are orthogonal if $|i-j| \ge 2$.

The ring R has an involution given by $X_i \mapsto X_i^{-1} = Y_i$. Since our form is to be skew Hermitian, we have that $h(\varepsilon_1, \varepsilon_1)$ is an element of R which is "imaginary" (i.e. it goes to its negative under the involution). We normalise it so that

$$h(\varepsilon_1, \varepsilon_1) = \frac{1 - X_1 X_2}{(1 - X_1)(1 - X_2)}.$$

Consider the element $h(\varepsilon_1, \varepsilon_2)$; the invariance of h under forces the equation

$$h(\varepsilon_1, \varepsilon_2) = h\big(s_1^2(\varepsilon_1), s_1^2(\varepsilon_2)\big).$$

Since $s_1^2(\varepsilon_1) = X_1 X_2 \varepsilon_1$, and $s_1^2(\varepsilon_2) = (1 - X_1)\varepsilon_1 + \varepsilon_2$, the invariance of h and the chosen value of $h(\varepsilon_1, \varepsilon_1)$ imply

$$h(\varepsilon_1, \varepsilon_2) = \frac{-X_2}{1 - X_2}, \qquad h(\varepsilon_2, \varepsilon_2) = \frac{1 - X_2 X_3}{(1 - X_2)(1 - X_3)}.$$

We can proceed in a like manner. Thus the $n \times n$-matrix $h(X) = (h_{ij})$ of the skew Hermitian form is

$$h(X) = \begin{pmatrix} \frac{1-X_1X_2}{(1-X_1)(1-X_2)} & -\frac{1}{1-X_2} & 0 & 0 & \dots & 0 \\ -\frac{X_2}{1-X_2} & \frac{1-X_2X_3}{(1-X_2)(1-X_3)} & -\frac{1}{1-X_3} & 0 & \dots & 0 \\ 0 & -\frac{X_3}{1-X_3} & \frac{1-X_3X_4}{(1-X_3)(1-X_4)} & -\frac{1}{1-X_4} & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Thus the invariance of h implies that h is the above matrix (once the value of $h(\varepsilon_1, \varepsilon_1)$ is normalised as above).

The natural formula for $h(X)$ given above takes values in the ring R' which is generated by R together with the inverse of the element $\prod_{i=1}^{n+1}(1-X_i)$. One can clear denominators and ensure that $h(X)$ takes values in R. Therefore $g_n(X)(P_{n+1}) \subset U(h(X))(S)$, where S is the sub-ring of R invariant under the involution $f \mapsto \overline{f}$ on R given by $X_i \mapsto X_i^{-1}$. The unitary group $U(h)$ is an affine algebraic group scheme defined over S:

$$U(h)(S) = \left\{g \in GL_n(R) : h(gv, gw) = h(v, w)\ \forall v, w \in R^n\right\}.$$

The following result is due to [2] (the skew Hermitian form in [2] is for a different basis):

Lemma 15

(1) *The skew Hermitian form on* (R^n) *defined by the matrix* $h(X)$ *is invariant under the action of the pure braid group. Therefore,* $\rho_X(P_{n+1}) \subset U(h)(S)$, *where* $S \subset R$ *is the sub-ring of elements invariant under the involution.*

(2) *The matrix* $h(X)$ *has determinant*

$$\frac{1 - X_1X_2\cdots X_nX_{n+1}}{(1-X_1)(1-X_2)\cdots(1-X_n)(1-X_{n+1})}.$$

(3) *In particular, the skew Hermitian form* h *is non-degenerate.*

Proof The form h was constructed under the assumption that it was invariant under P_{n+1}. Hence we need only prove part (2) of the lemma. Part (2) of the lemma is proved by induction. Put $X = (X_1, X')$ where X' is the n-tuple $(X_2, \ldots, X_{n+1})$; write $X' = (X_2, X'')$ where X'' is the $n-1$-tuple $(X_3, X_4, \ldots, X_{n+1})$. Expand the determinant of the $n \times n$ matrix $h_n(X) = h(X)$ by the first row; then $h_n(X)$ may be expressed in terms of $h_{n-1}(X')$ and $h_{n-2}(X'')$:

$$h_n(X) = \frac{1 - X_1X_2}{(1-X_1)(1-X_2)}h_{n-1}\big(X'\big) - \frac{X_2}{(1-X_2)^2}h_{n-2}\big(X''\big).$$

Now induction on n implies the formula for the determinant of $h_n(X)$. $\square$

Suppose that $\mathfrak{a} \subset R$ is a non-zero ideal invariant under the involution $f \mapsto \overline{f}$ on R. Then the quotient map $R \to R/\mathfrak{a}$ induces a homomorphism $GL_n(R) \to GL_n(R/\mathfrak{a})$; the skew Hermitian form h descends to a skew Hermitian form $h_{\mathfrak{a}}$ on the quotient module $(R/\mathfrak{a})^n$ and hence we have a homomorphism $U(h)(S) \to U(h_{\mathfrak{a}})(S/\mathfrak{a} \cap S)$; therefore we have a representation $g_{n,\mathfrak{a}} : P_{n+1} \to U(h_{\mathfrak{a}})(S/\mathfrak{a} \cap S)$.

Now consider a homomorphism from the ring $R = \mathbb{Z}[X_i^{\pm 1}; 1 \le i \le n+1]$ into the ring $\mathbb{Z}[\omega_d]$ of integers in the cyclotomic extension $E_d = \mathbb{Q}(e^{\frac{2\pi i}{d}}) = \mathbb{Q}(\omega_d)$ (ω_d is the primitive d-th root of unity $e^{\frac{2\pi i}{d}}$). This homomorphism is given by $X_i \mapsto t_i$ where $t_i = \omega_d^{k_i}$ is a d-th root of unity. Since all the k_i are co-prime to d, it follows that the group generated by each of the t_i is all of $\mathbb{Z}/d\mathbb{Z}$, the group of d-th root of unity. Under the homomorphism $R \to \mathbb{Z}[\omega_d]$, the sub-ring S maps into the ring O_d of integers in the totally real sub-field $\mathbb{Q}(2\cos\frac{2\pi}{d})$ of the cyclotomic field E_d. Hence we have the composite representation, denoted

$$g_n(k,d) : P_{n+1} \to U(h)(S) \to U(h)(O_d),$$

where k is the $(n+1)$-tuple of integers $(k_1, k_2, \dots, k_{n+1})$.

The following is the main result of the paper, from which Theorem 1 will be deduced.

Theorem 16 *Suppose $d \ge 3$, $n \ge 2d$ and all the integers k_i are co-prime to d. Then the image $\Gamma_n = \Gamma = g_n(k,d)(P_{n+1})$ (of the Gassner representation $g_n(k,d)$ at primitive d-th roots of unity) is a subgroup of finite index in the integral unitary group $U(h)(O_d)$. In other words, the "monodromy group" $g_n(k,d)(P_{n+1})$ is an arithmetic group.*

The full braid group B_{n+1} has a representation, called *the reduced Burau representation* ([5], p. 118, Example 3),

$$\rho_n(q) : B_{n+1} \to GL_n\big(\mathbb{Z}[q,q^{-1}]\big).$$

Since the restriction to P_{n+1} of the reduced Burau representation at primitive d-th roots of unity is the reduced Gassner representation $g_n(k,d)$ evaluated at primitive d-th roots of unity (when all the k_i are equal to 1), the following Theorem is a special case of Theorem 16.

Theorem 17 *If $d \ge 3$ and $n \ge 2d$ and all the k_i are 1, then the image of the Burau representation*

$$\rho_n(d) : B_{n+1} \to U(h)(O_d)$$

evaluated at all primitive d-th roots of unity, is a subgroup of finite index. In other words, the monodromy group $\rho_n(d)(B_{n+1})$ is an arithmetic group.

Theorem 17 was proved in [21], by using properties of the Burau representation at roots of unity, and by using induction for all $n \ge 2d$. The proof of Theorem 16 is similar, and we use properties of the reduced Gassner representations at roots of unity. These properties are essentially well known, but we need a precise form of these results. Theorem 16 will be proved at the end of this section, after many preliminary results.

4.2 Irreducibility

As we have seen before, the reduced Gassner representation has a nondegenerate invariant skew Hermitian form h with values in the field of fractions Ω of the ring $R=\mathbb{Z}[X_1^{\pm 1},\dots,X_{n+1}^{\pm 1}]$ of Laurent polynomials, which is preserved by the group P_{n+1} under the Gassner representation. This was determined on the basis ε_i in Sect. 4.1. By Lemma 14, it follows that the elements s_i^2 are complex reflections.

Proposition 18

(1) *If* $R=\mathbb{Z}[X_1^{\pm 1},X_2^{\pm 1},\dots,X_{n+1}^{\pm 1}]$, *and* Ω *is its quotient field, then the reduced Gassner representation*

$$G_n(X): P_{n+1}\to GL_n(R)\subset GL_n(\Omega),$$

is irreducible.

(2) *The central element* $\Delta^2\in P_{n+1}$ *where*

$$\Delta=\Delta_n=(s_1s_2\cdots s_n)(s_1s_2\cdots s_{n-1})\cdots(s_1s_2)(s_1),$$

acts by multiplication by the scalar $X_1X_2\cdots X_{n+1}$ *on the reduced Gassner representation.*

(3) *If* $W\subset R^n$ *is an additive subgroup stable under the action of* P_{n+1} *then there exists a scalar* $\lambda\in R$ *such that* $\lambda(R^n)\subset W$.

Proof (1) At the beginning of this subsection, we have verified that the conditions of Lemma 7 are satisfied. By the result of [10] quoted earlier, Ω^n admits a nondegenerate Hermitian form. Therefore, by Lemma 7, the representation $g_n(X)$ is absolutely irreducible.

(2) The element Δ^2 is central in P_{n+1}; by part (1), the central element acts by a scalar, call it λ. Write π_{n+1} for the product $X_1X_2\cdots X_{n+1}$. We compute the scalar λ by finding the effect of Δ^2 on the element e_1. Consider the element $x_1\in F_{n+1}$. A calculation (see the formulae for the action of s_i on the free group F_{n+1} in Sect. 3.2), shows that the action of Δ^2 on x_1 is given by (notation: in a group, ${}^{y}(x)=yxy^{-1}$)

$$\Delta^2(x_1)={}^{x_1x_2\cdots x_{n+1}}(x_1).$$

In the semi-direct product $K^{ab} \rtimes F_{n+1}$, the element x_i maps to $\overline{x}_i = (e_i, X_i) = e_i X_i$. Hence this equation then becomes

$$\begin{aligned}\Delta^2(e_i, X_i) = \Delta^2(e_1 X_1) &= (v\pi_{n+1})(e_1 X_1)\pi_{n+1}^{-1} v^{-1} \\ &= \big(\pi_{n+1} e_1 + (1 - X_1)v, X_1\big).\end{aligned}$$

Comparing the vector parts, we get

$$\Delta^2(e_1) = X_1 X_2 \cdots X_{n+1}(e_1) + (1 - X_1)v.$$

This is in the Gassner representation space K^{ab}. Going modulo the line through v, we see that

$$\Delta^2(e_1) = \pi_{n+1} e_1 = X_1 X_2 \cdots X_{n+1} e_1 \pmod{Rv}$$

in the reduced Gassner representation. This proves part (2) of the proposition.

We will now prove part (3). Fix i with $1 \le i \le n$. The group P_{i+1} operates on the R module generated by $\varepsilon_1, \dots, \varepsilon_i$. This module is nothing but the reduced Gassner representation $g_i(X)$. Consequently, the element $C_i = (\Delta_i^2)$ acts by the scalar $X_1 \cdots X_{i+1}$ on the vectors $\varepsilon_1, \dots, \varepsilon_i$. Moreover, $s_i^2(\varepsilon_i) = X_i X_{i+1}(\varepsilon_i)$. In particular, the group H_i generated by the elements $X_1 \cdots X_{i+1}, X_1 \cdots X_{i+2}, \dots, X_1 \cdots X_{n+1}; X_i X_{i+1}$ has the property that the Γ module generated by ε_i contains all elements of the form $h(\varepsilon_i)$ for every $h \in H_i$. Note that H_i is also the group generated by the elements $X_1 \cdots X_{i+1}, X_{i+2}, \dots, X_{n+1}, X_i X_{i+1}$ (successive ratios of the previous set of elements, together with the first and the last one of the previous set of elements).

Thus the group H generated by $H_1, \dots, H_n$ is the group generated by $X_1 X_2, X_3, \dots, X_{n+1}$ (contribution from $i = 1$) and $X_2 X_3$ (contribution from $i = 2$). This is clearly the group generated by $X_1, \dots, X_{n+1}$. By Corollary 2, this means that there exists a $\lambda \neq 0$ in R such that $\lambda X_1^{\mathbb{Z}} \cdots X_{n+1}^{\mathbb{Z}}(\varepsilon_i)$ lies in the Γ-module generated by all the ε_i. Since monomials in the X_i generate R, this means that $\lambda(R\varepsilon_j) \subset \mathbb{Z}[\Gamma]\varepsilon_i$ for all i, j.

Now, the additive group W stable under the action of $\Gamma = P_{n+1}$ has the property that it contains a non-zero scalar multiple $\mu(\varepsilon_i)$ for all i by Lemma 7. Hence $\lambda\mu(R\varepsilon_i) \subset W$. This is (3) of the proposition. □

Notation Suppose $\mathfrak{a}$ is a prime ideal in R invariant under the involution on R given by $X_i \mapsto X_i^{-1}$ for each i. Consider the integral domain $A = R/\mathfrak{a}$. Let B denote the invariants in A under the involution. We get the corresponding reduced Gassner representation $g_n(A)$ on the module $\bigoplus_{i=1}^n A\varepsilon_i$ of P_{n+1}. Write t_i for the image of X_i under the quotient map $R \to A = R/\mathfrak{a}$. The skew Hermitian form h reduced modulo $\mathfrak{a}$ gives a skew Hermitian form on

A^n and the image of P_{n+1} under $g_n(A)$ lies in $U(h)(A)$. Let E denote the quotient field of A. Then $W(A) = \bigoplus A\varepsilon_i$ is a subgroup of the E-vector space $W(E) = \bigoplus E\varepsilon_i$.

Proposition 19

(1) *If $t_1 \cdots t_{n+1} \neq 1$ then $h(A)$ is non-degenerate and the representation $g_n(A)$ is irreducible.*
(2) *The central element Δ_n^2 of P_{n+1} [where $\Delta_n = (s_1 \cdots s_n) \cdots (s_1 s_2) s_1$] acts by the scalar $t_1 \cdots t_{n+1}$ on the representation $g_n(A)$.*
(3) *If $t_1 \cdots t_{n+1} \neq 1$ then every nonzero P_{n+1}-invariant subgroup of $A^n = \bigoplus A\varepsilon_i$ contains $\lambda(A^n)$ for some non-zero element $\lambda \in A$.*

The proof of Proposition 18 can be repeated for the quotient $R/\mathfrak{a}$ in place of R.

5 Proof of Theorem 16

We will first prove some preliminary results on the reduced Gassner representation at d-th roots of unity. We will then use induction to deduce Theorem 16 from these results.

5.1 The reduced Gassner representation at roots of unity

Consider the reduced Gassner representation $g_n(k) : P_{n+1} \to GL_n(R)$ where R is the Laurent polynomial ring in $n+1$ variables X_i with integral coefficients and R^n is the free module $W_n(X) = \bigoplus_{i=1}^n R\varepsilon_i$. We now specialise to $X_i \mapsto t_i = t^{k_i}$. The resulting representation is from P_{n+1} into $GL_n(A_d) \subset GL_n(E_d)$ where E_d is the d-th cyclotomic extension of $\mathbb{Q}$ and A_d the ring of integers in E_d, and is denoted $g_n(k,d)$. Denote the vector space $W_n(k,d)$.

Lemma 20 *The reduced Gassner representation evaluated at all primitive d-th roots of unity is irreducible if and only if $t_1 t_2 \cdots t_{n+1} \neq 1$.*

When $t_1 \cdots t_{n+1} = 1$, the representation space $W_n(k,d)$ of the representation $g_n(k,d)$ contains a non-zero invariant vector w and the restriction of the quotient representation $W_n(k,d)/E_d w$ to the subgroup P_n is isomorphic to $g_{n-1}(k,d)$. If we denote the quotient representation by $\overline{g}_n(k,d)$, then $\overline{g}_n(k,d)$ is irreducible.

If we denote by π_i the product $t_1 t_2 \cdots t_i$, then the invariant element $w \in W_n(k,d)$ is given by

$$w = \sum (1 - \pi_i)\varepsilon_i.$$

Proof If $t_1 \cdots t_{n+1} \neq 1$, then the specialisation of the Hermitian form h at $t_1, \ldots, t_{n+1}$ has non-zero determinant, by Lemma 15. Therefore, by Lemma 7, the representation $g_n(k,d)$ is irreducible.

Suppose $t_1 t_2 \cdots t_{n+1} = 1$. Consider the element

$$v = e_1 + t_1 e_2 + \cdots + t_1 t_2 \cdots t_{n-1} e_n + t_1 t_2 \cdots t_{n+1} e_{n+1}.$$

This is the vector part of the P_{n+1} invariant element $x_1 x_2 \cdots x_{n+1}$ in the semi-direct product $K^{ab} \rtimes t^{\mathbb{Z}}$, and is hence invariant. The expression for v in terms of ε_i and v_{n+1} shows that

$$v = \sum_{i=1}^{n} (1 - t_1 t_2 \cdots t_i)\varepsilon_i + (1 - t_1 \cdots t_{n+1}) v_{n+1}.$$

By assumption on the t_i, the coefficient of v_{n+1} is zero, and hence v lies in the reduced Gassner representation $W_n(k,d)$ (the span of the ε_i).

Let P_n be the set of pure braids in the group generated by $s_2, s_3, \ldots, s_n$. The module $W = W_n(k,d)$ is spanned by the vector v and $\varepsilon_2, \ldots, \varepsilon_n$. Therefore, $g_n(k,d)$ restricted to P_n splits into a direct sum of the modules Rv and $g_{n-1}(k,d)$; the latter is irreducible by part [1], since $t_2 t_3 \cdots t_{p+1} = t_1^{-1} \neq 1$. We have therefore proved that the representation $\overline{g_n(k,d)}$ restricted to P_n is the representation $g_{n-1}(k,d)$ and is irreducible for the subgroup P_n; hence it is irreducible for the bigger group P_{n+1}. □

The lemma is due essentially to Abdulrahim [1] and [2]. We have derived the lemma since we need the explicit formula for the invariant element, in terms of the basis ε_i.

5.2 Gassner representations with degenerate Hermitian forms

Suppose that t_i are all primitive d-th roots of unity and that $t_1 t_2 \cdots t_p = 1$. Consider the basis $\varepsilon_1, \ldots, \varepsilon_{p-1}$ of W. This contains the element

$$w = \sum_{i=1}^{p-1} (1 - t_i)\varepsilon_i.$$

Therefore, it is part of a basis of W: $w, \varepsilon_2, \ldots, \varepsilon_{p-1}$. Since w is invariant, with respect to this basis, every element g of the pure braid group P_p has the matrix form

$$\begin{pmatrix} 1 & v \\ 0_{p-2} & \alpha \end{pmatrix}$$

where α is the matrix of g acting on the quotient $W/E_d w$ with respect to the basis $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{p-1}$. The element $C' = (\Delta')^2$ where

$$\Delta' = (s_2 \cdots s_p)(s_2 \cdots s_{p-2}) \cdots (s_2 s_1)(s_2),$$

is central in the pure braid group P_{p-2} and hence acts by a scalar on the irreducible representation; the scalar is $t_2 t_3 \cdots t_p = t_1^{-1} \neq 1$ (by part (2) of Proposition 19). Therefore, the commutator

$$u = \left[g, \left(\Delta'\right)^2\right] = \begin{pmatrix} 1 & v' \\ 0_{p-2} & 1_{p-2} \end{pmatrix}$$

acts by an upper triangular unitary matrix.

Proposition 21 *Let $t_1, \ldots, t_p$ be primitive d-th roots of unity with $t_1 \cdots t_p = 1$. Consider the representation $g(p,t) : P_p \to U(h)(O_d)$, the reduced Gassner representation $g(p, X)$ specialised to $X_i \mapsto t_i$. Then*

(1) *The unitary group $U(h)$ has a unipotent radical isomorphic to the P_{p-1} module W^*, where W is the reduced Gassner representation $g(p-1, t)$.*
(2) *If $g = s_1^2$ and Δ' is as in the preceding paragraph, then the image of the element $u = [g, (\Delta')^2]$ under the reduced Gassner representation $g(p,t)$ is not identity.*
(3) *The conjugates $\{huh^{-1}\}$ of u for $h \in P_{p-1}$ generate a subgroup of finite index in the integral additive subgroup $W^*(O_d)$ of W^*.*

Proof Part (1) is obvious.

The vector

$$w = \sum_{i=1}^{p} (1 - t_1 \cdots t_i)\varepsilon_i$$

(notice that the coefficient of ε_p is zero) is orthogonal to all the vectors $\varepsilon_1, \ldots, \varepsilon_p$ and is invariant under all of P_p. We can consider the basis B' given by $w, \varepsilon_2, \ldots, \varepsilon_p$ of the vector space $W = \bigoplus_{i=1}^p \varepsilon_i$. Now s_1^2 fixes w. By Lemma 14, we have the equalities $s_1^2(\varepsilon_2) = \varepsilon_2 + (1 - t_1)\varepsilon_1 = \varepsilon_2 + w - (1 - t_1 t_2)\varepsilon_2 - \cdots - (1 - t_1 \cdots t_p)\varepsilon_p$, and $s_1^2(\varepsilon_i) = 1$ for $i \geq 3$. Therefore the matrix of s_1^2 in this basis B' is of the form

$$g(p,t)\left(s_1^2\right) = \begin{pmatrix} 1 & 1 & 0 & \ldots & 0 \\ 0 & t_1 t_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

Consider the pure braid group P'_{p-1} generated by $s_2^2, \ldots, s_p^2$. The subspace spanned W' by $\varepsilon_2, \ldots, \varepsilon_p$ is left stable under P'_{p-1} and the resulting representation is the Gassner representation for P'_{p-1}. Consequently, the element $(\Delta')^2$ acts by the scalar $c = t_2 \cdots t_p = t_1^{-1} \neq 1$ on W'. Moreover, $(\Delta')^2$ fixed the vector w. Hence with respect to the basis B' the element $(\Delta')^2$ has the matrix

$$g(p,t)\big((\Delta')^2\big) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & c & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & c \end{pmatrix}.$$

It is then clear from the above matrix forms that the commutator of s_1^2 and $(\Delta')^2$ has the matrix form

$$g(p,t)(u) = g(p,t)\big(\big[s_1^2, (\Delta')^2\big]\big) = \begin{pmatrix} 1 & t_1^{-1}t_2^{-1}(1-c^{-1}) & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \end{pmatrix}.$$

In other words, this lies in the vector group $W^*(O_d)$ and is a non-identity element (since $c \neq 1$). This proves part (2).

Consider the image (under $g(p,t)$) of the group generated by the conjugates huh^{-1} of u by elements h (of the smaller braid group P_{p-1} generated by $s_2, s_3, \ldots, s_{p-1}$). This image may be identified with an additive subgroup A of the vector group W'^* where W'^* is dual of the reduced Gassner representation W' at d-th roots of unity for P_{p-1}. It follows from part [3] of Proposition 19 that the additive subgroup A contains a subgroup of finite index in the (dual of the) vector group

$$W'(A_d) \simeq \bigoplus_{i=2}^{p} A_d \varepsilon_i.$$

□

We now consider the "next" pure braid group P_{p+1}, with $t_1 t_2 \cdots t_p = 1$. The Gassner representation takes P_{p+1} into a unitary group $U(h)$. The product $t_1 t_2 \cdots t_{p+1} = t_{p+1} \neq 1$ since all the t_i are primitive d-th roots of unity. With respect to the basis $w, \varepsilon_2, \ldots, \varepsilon_{p-1}$ and ε_p, the matrix of the commutator $u = [s_1^2, (\Delta')^2]$ now takes the form

$$u = \begin{pmatrix} 1 & v & \lambda \\ 0_{p-2} & 1_{p-2} & \overline{v} \\ 0 & 0 & 1 \end{pmatrix},$$

and the group generated by the conjugates $\{mum^{-1} : m \in P_{p-1}\}$ is a subgroup of finite index in the Heisenberg group $H(X)$ where $H(X)$ is the subgroup of $U(h)$ which acts unipotently on the flag

$$E_d w \subset E_d w + X \subset V,$$

where X is the span of $\varepsilon_2, \ldots, \varepsilon_{p-1}$, and V is the span of $\varepsilon_1, \ldots, \varepsilon_p$ (the reduced Gassner representation). Hence the Gassner image contains a subgroup of finite index in the unipotent integral Heisenberg group $H(X)(O_d)$.

Proposition 22 *Let $t_1, t_2, \ldots, t_p, t_{p+1}$ be primitive d-th roots of unity such that $t_1 t_2 \cdots t_p = 1$. Denote by Γ_p the image of the Gassner representation*

$$g_p(k, d) : P_{p+1} \to U(h)(O_d) \subset GL_p(A_d).$$

Then

(1) *there exist two opposite maximal parabolic subgroups P and P^- of $U(h)$ such that the image Γ_p intersects the integral unipotent radicals $U_P(O_d)$ and $U_P^-(O_d)$ in subgroups of finite index.*
(2) *In particular, if $K - rank(U(h)) \geq 2$, then the image Γ_p is an arithmetic subgroup of (i.e. subgroup of finite index in) $U(h)(O_d)$.*

Proof Since $t_1 t_2 \cdots t_{p+1} = t_{p+1} \neq 1$, it follows that the Hermitian form $h = h_p$ is non-degenerate. Therefore, the unitary group $U(h)$ is reductive. Denote by $V = V_p = E_d^p$ the natural representation of $U(h)$ (E_d is the d-th cyclotomic extension of $\mathbb{Q}$).

We have already seen in the paragraph preceding the statement of the proposition, that Γ_p contains a subgroup U_0 of finite index in the integer points of the Heisenberg group $H(X)$. Recall that $H(X)$ is the unipotent radical of a parabolic subgroup P (i.e. P is the normaliser of $H(X)$ in the unitary group $U(h)$). We will now prove that there exists a conjugate of U_0 in Γ_p which is an arithmetic subgroup of an opposite unipotent radical.

By assumption, there exists an isotropic vector $v = \sum_{i=1}^{p-1}(1 - \pi_i)\varepsilon_i$ (it is orthogonal to all the ε_i with $i \leq p-1$ and is therefore orthogonal to itself). Consequently, we may find a basis

$$v = w_1, \ldots, w_r, x_1, \ldots, x_s, w_1^*, \ldots, w_r^*$$

of V where x_i are orthogonal to the w_i and w_i^*, w_i are isotropic mutually orthogonal vectors, similarly, w_j^* are mutually orthogonal isotropic vectors, and $w_j^*(w_i) = \delta_{ij}$ (the Kronecker delta symbol). The intersection of the diagonals with $U(h)$ then gives a maximal torus defined over K, which is maximally K-split over K. $N(T)$ denotes the K-Weyl group and κ denotes the longest element in the K-Weyl group.

The following Zariski density statement is very likely true in greater generality (cf. Lemma 11.5 of [6] and Lemma 4.4 of [11]), but we will need only this weaker version in the course of the proof.

Lemma 23 *Suppose that $\Gamma \subset U(h)(O_d)$ is the image of the pure braid group under the reduced Gassner representation. Suppose that Γ_n contains a finite index subgroup of the Heisenberg group which can be viewed as the integral unipotent radical of a* (*maximal*) *parabolic K_d-subgroup of $U(h)$ and that Γ acts irreducibly on the reduced Gassner representation* (*i.e. suppose that $t_1t_2\cdots t_{n+1} \neq 1$*). *Then, the Zariski closure of Γ_n contains the special unitary group $SU(h)$.*

Proof We will use Proposition 8 of Sect. 2. The irreducibility of the action of Γ_n implies that the Zariski closure (intersected with $SL_n(\mathbb{C})$) is reductive. Since Γ_n is assumed to contain a finite index subgroup of the group of integral points of the unipotent radical given by the Heisenberg group, it follows that if H is the Zariski closure of Γ_n then H contains U, the group of the form in Proposition 8. Then Proposition 8 implies the lemma. □

We continue with the proof of Proposition 22. By Lemma 23, it follows that Γ_p intersects the big Bruhat cell (the Zariski open set $U\kappa P$ where κ is the longest Weyl group element above). Let γ lie in the image Γ_p and also in the big Bruhat cell. Then $\gamma P\gamma^{-1} = P'$ is opposite to P and hence $\gamma U_0\gamma^{-1}$ lies in Γ_p. Therefore, the first part of the proposition follows.

To prove the second part, note that $U(h)(O_d)$ is a higher rank lattice containing Γ. By part (1) of the proposition, Γ contains a finite-index subgroup of integral points of the unipotent radical of a parabolic K-subgroup. By Lemma 23, Γ is Zariski dense. Hence by Theorem 4, the second part follows. □

5.3 Proof of Theorem 16

Recall the notation: let $k_1, k_2, \ldots, k_{n+1}$ be integers with $1 \leq k_i \leq d-1$ and co-prime to d. Let $\omega_d = e^{2\pi i/d}$ be a primitive d-th root of unity; write $t_i = \omega_d^{k_i}$. Let $\mathbb{Q}(\omega_d)$ be the d-th cyclotomic extension and A_d the ring of integers in E_d; denote by K_d the maximal totally real sub-field $\mathbb{Q}(2\cos(2\pi/d))$ of E_d and O_d the ring of integers in K_d. Denote by $g_n(k,d): P_{n+1} \to U(h)(O_d) \subset GL_n(A_d)$ the reduced Gassner representation specialised at $X_i \mapsto t_i$. In this section, for ease of notation (since we use induction and have to deal with many indices) we denote by V_n the E_d vector space spanned by $\varepsilon_1, \ldots, \varepsilon_n$ (this is the same as $W_n(k,d)$).

Suppose $s \in B_{n+1}$ is an element of the *full* braid group, whose image in S_{n+1} is the permutation σ. Now σ operates on the Laurent polynomial ring

$\mathbb{Z}[X_1^{\pm 1}, \ldots, X_{n+1}^{\pm 1}]$ by permutations of the X_i's. Given s, denote by u the $n \times n$ matrix (u_{ij}) such that

$$\sigma(\varepsilon_i) = \sum_{j=1}^{n} u_{ji} \varepsilon_j.$$

Recall that B_{n+1} operates on the Gassner module $W \otimes \Omega$ by automorphisms of the Abelian group, *but not linearly* over Ω. Let $s \in B_{n+1}$, σ its image in S_{n+1}. It easily follows from the construction of the Gassner representation that, for all g in the pure braid group P_{n+1}, we have the equality

$$g(n, X)\big(sgs^{-1}\big) = \sigma(u)\sigma(g(n, X)(g)\sigma(u)^{-1}.$$

In particular, the image of $g(n, X)$ is a conjugate, by an element in $GL_n(R)$, of the image of the twisted representation $\sigma(g(n, X)) = g(n, \sigma(X))$. Therefore, if we wish to prove that the specialisation of $g(n, X)$ at some roots of unity is arithmetic, it is enough to prove it for the "twisted" representation $\sigma(g(n, X))$ (the notation $\sigma(g(n, X))$ means that the matrix entries of $g(n, X)$ which are elements of R acted upon by the permutation σ of the variables X_i).

We now begin the proof of Theorem 16.

Proof We first prove that there exist two elements $v, v' \in E_d^n$ which are linearly independent and mutually orthogonal with respect to the Hermitian form.

Consider the d numbers $t_1, t_1 t_2, \ldots, t_1 t_2 \cdots t_{d-1}$ and $t_1 t_2 \cdots t_d$. These are d elements of the group μ_d of d-th roots of unity. Therefore, by the pigeon-hole principle, we have two possibilities: (a) one of these products is one, or (b) two of them coincide.

Hence there exists a subset $I \subset \{1, 2, \ldots, d-1, d\}$ consisting of l consecutive integers such that $\prod_{i \in I} t_i = 1$. Let $I = \{a+1, \ldots, a+l\}$. Put

$$v = \sum_{i=1}^{l-1} (1 - t_{a+1} \cdots t_{a+i}) \varepsilon_{a+i}.$$

By an earlier computation, v is orthogonal to all the ε_{a+i} with $1 \le i \le a+l$. Note that in the expression of v as a linear combination of the ε_i, the "last" basis vector ε_{a+l} does not appear.

Similarly, there exists a subset $J \subset \{d+1, \ldots, 2d\}$ consisting of m consecutive numbers such that $\prod_{j \in J} t_j = 1$; let $J = \{b+1, \ldots, b+m\}$, where $b \ge l+1$. As before,

$$v' = \sum_{j=1}^{m-1} (1 - t_{b+1} \cdots t_{b+j}) \varepsilon_{b+j},$$

is isotropic and is orthogonal to all the ε_{b+j} with $1 \le j \le m$. Since the indices of the ε_μ occurring in v are of the form $\mu = a + i$ with $i \le l - 1$, it follows that $a + i \le d - 1$. Since the indices of $\varepsilon_\nu = \varepsilon_{b+j}$ occurring in the expression for v' are of the form $\nu = b + j \ge d + 1$, it follows that $\nu - \mu \ge 2$. Therefore, v, v' are orthogonal.

Therefore the E_d-rank of the span of the vectors $\varepsilon_{a+1}, \dots, \varepsilon_{a+l}$ and $\varepsilon_{b+1}, \dots, \varepsilon_{b+m}$ is at least two.

By the remarks preceding the beginning of the proof of Theorem 16, we may assume that $a = 0$ and $b = l$, after a permutation σ of the indices, so that $t_1 \cdots t_l = 1$ and $t_{l+1} \cdots t_{l+m} = 1$. Therefore $t_1 \cdots t_{l+m+1} = t_{l+m+1} \ne 1$ and V_n is non-degenerate if $n = l + m$. By Proposition 22, the group Γ_{l+m} intersects two opposite integral unipotent radicals U_P^+ and U_P^- in subgroups of finite index. By the conclusion of the preceding paragraph we get that for $n = l + m$, the group $U(h)$ has K-rank at least two, and therefore the group Γ_n is arithmetic, by Theorem 4.

If the theorem is true for *some* $n \ge l + m$, then we will prove that it is true for $n + 1$. There are several cases to consider.

Let $V = V_n$ (resp. V_{n+1}) be the span of $\varepsilon_1, \dots, \varepsilon_n$ (resp. $\varepsilon_1, \dots, \varepsilon_{n+1}$). Let V_n' be the span of $\varepsilon_2, \dots, \varepsilon_{n+1}$. Then, the intersection $V_n \cap V_n'$ is the span of $\varepsilon_2, \dots, \varepsilon_n$. Therefore, $V_n \cap V_n'$ contains a subspace W'' which is non-degenerate and contains an isotropic vector (e.g. take W'' to be the span of $\varepsilon_l, \varepsilon_{l+1}, \dots, \varepsilon_{l+m+1}$).

Case 1. Assume that V_n, V_n', V_{n+1} are all non-degenerate. Since $n \ge l + m$ the K-rank of $U(h_{n+1})$ and $U(h_n)$ are both ≥ 2. By induction assumption, $U(V_n) \cap \Gamma$ is arithmetic and $U(V_n') \cap \Gamma$ is arithmetic. By Lemma 5, $U(h_{n+1})$ is also arithmetic.

Case 2. V_{n+1} is non-degenerate but V_n is degenerate. Then by Proposition 22, $U(h_{n+1}) \cap \Gamma$ is arithmetic. Similarly, if V_{n+1} non-degenerate but V_n' is degenerate, Γ_{n+1} can be proved to be arithmetic.

Case 3. V_{n+1} is degenerate. Then V_{n+1} contains a one dimensional null space $E_d v$ and by induction, the image of Γ in $U(V_{n+1}/Av)$ is arithmetic. However, by part (3) of Proposition 21, the group Γ intersects the integral unipotent radical of $U(V_{n+1})$ in a finite index subgroup. Therefore, Γ contains a finite index subgroup of the integral unipotent radical of $U(h)$ and maps onto a finite index subgroup of the reductive Levi part of $U(h)(O_d)$. Hence Γ is a subgroup of finite index.

If $n \ge 2d$, then in the above notation, $n \ge l + m$. Now the three induction steps proved above imply that the group Γ_n is arithmetic. □

6 Homology of cyclic coverings

In this section, we will view the first $\mathbb{Q}$-homology of certain index d subgroups of the free group on $n + 1$ generators, with essentially a direct sum of the reduced Gassner representation evaluated at d-th roots of unity. The proof is a little indirect, since it does not seem possible to get a natural basis of the homology of index d-subgroup, which generates the relevant Gassner representation. Instead, we replace the free group F_{n+1} on $n+1$ generators with a free product of F_{n+1} and an auxiliary $\mathbb{Z}$ (the free product is then the free group on $n+2$ generators). It seems easier to identify the homology of a finite index subgroup of the latter, with a specialisation of the Gassner representation.

6.1 Image of homology

Let $k_1, k_2, \dots, k_{n+1}$ be integers co-prime to d with $1 \le k_i \le d-1$. Write $k = (k_1, k_2, \dots, k_{n+1})$. We get a homomorphism from the free group $F_{n+1} \to \mathbb{Z}/d\mathbb{Z}$ (the latter written multiplicatively as $q^{\mathbb{Z}}/q^{d\mathbb{Z}}$) by sending x_i to the element q^{k_i}. Denote by $K(k,d)$ the kernel to this map. We have thus an exact sequence

$$1 \to K_0(k,d) \to F_{n+1} \to \mathbb{Z}/d\mathbb{Z} \to 1.$$

Being a subgroup of finite index (in fact of index d) in F_{n+1}, the group $K_0(k,d)$ is also free on n_0' generators, with

$$\left(1 - n_0'\right) = d\left(1 - (n+1)\right), \quad \text{i.e.} \quad n_0' = 1 + nd.$$

Since the k_i are co-prime to d, the image of x_i generates $\mathbb{Z}/d\mathbb{Z}$; hence there exists an element $\xi \in F_{n+1}$ such that its image is q. Then the elements $z_i = x_i \xi^{-k_i}$ lie in the kernel $K_0(k,d)$; moreover, the elements ξ and $\{z_i : 1 \le i \le n+1\}$ generate F_{n+1}. Hence the z_i and ξ^d generate the kernel $K_0(k,d)$ as a normal subgroup of F_{n+1}: if we go modulo the normal subgroup generated by the z_i, then the resulting group is generated by the image of ξ and maps onto $\mathbb{Z}/d\mathbb{Z}$.

The first homology group $V_0 = K_0(k,d)^{ab} \otimes \mathbb{Q}$ with $\mathbb{Q}$ coefficients of $K_0(k,d)$ is therefore a vector space of dimension $n_0' = 1 + nd$ over $\mathbb{Q}$. We have already seen that this homology group is a module over the quotient group $\mathbb{Z}/d\mathbb{Z}$, and is hence a module over the group ring $\mathbb{Q}[q]/(q^d - 1)$. As a $\mathbb{Q}$ vector space, V_0 is generated by the elements $\xi^j(z_i)$: ($2 \le i \le n+1$ and $0 \le j \le d-1$) and the element ξ^d. Hence these elements form a basis of the first homology group over $\mathbb{Q}$.

We first find the invariants V_0^G in V_0 under the action of the group $G = \mathbb{Z}/d\mathbb{Z}$. Since the image q of ξ generates G, it follows that $V_0 = (V_0/(1 + q + \cdots + q^{d-1})) \oplus V_0^G$. In this decomposition, the element q may be replaced by q^{k_i} for any i since k_i is coprime to d.

We know that $\xi^d \in V_0$ is invariant. Moreover, $x_i^d \in V_0$ is invariant under conjugation by x_i; but the conjugation action of F_{n+1} on V_0 descends to that of G and each x_i generates G. Therefore, x_i^d is invariant under G. We write x_i^d in terms of the $x_j = z_i\xi^{k_i}$:

$$x_i^d = z_i\bigl(1 + q^{k_i} + \cdots + q^{k_i(d-1)}\bigr).$$

As an operator on V_0 multiplication by the element $M_i = 1 + q^{k_i} + \cdots + q^{(d-1)k_i}$ is zero on non-invariants and d on invariants. Therefore, M_i is independent of k_i and hence $x_i^d = z_i(1 + q + \cdots + q^{d-1})$. In other words the $\mathbb{Q}$-span of the x_i^d is the space of all invariants V_0^G:

$$V_0^G = \sum_{i=1}^{n+1} \mathbb{Q}\bigl[x_i^d\bigr]. \tag{11}$$

Consider the quotient $V_0^{ni} = V_0/(1 + q + \cdots + q^{d-1})V_0$ (ni stands for non-invariants). This is a module over the quotient ring $\mathbb{Q}[q]/(1 + q + \cdots + q^{d-1})$. By the discussion of the preceding paragraphs, the first homology V_0 is generated as a $\mathbb{Q}[q]/(q^d - 1)$ module by (the images in the Abelianisation of $K_0(k, d)$ of) the elements z_i and by ξ^d. Since ξ commutes with ξ^d it follows that in the homology group, $q(\xi^d) = \xi^d$; hence the augmentation ideal of the group ring $\mathbb{Q}[q]/(q^d - 1)$ kills ξ^d. Therefore we see that the z_i form a basis of the free module V_0^{ni} as a module over the ring $R_d = \mathbb{Q}[q]/(1 + q + \cdots + q^{d-1})$:

$$V_0^{ni} = R_d^n.$$

We also have the free group on $n + 2$ generators written $F_{n+1} * t^{\mathbb{Z}}$, with a natural inclusion of F_{n+1} in F_{n+2}. On F_{n+2} we have a homomorphism into $\mathbb{Z}/d\mathbb{Z}$ by sending x_i to q^{k_i} and t to the standard generator q. Denote by $K(k, d)$ the kernel to this map; we have a short exact sequence

$$1 \to K(k, d) \to F_{n+1} * t^{\mathbb{Z}} \to \mathbb{Z}/d\mathbb{Z} \to 1.$$

For similar reasons, $K(k, d)$ is free on n' generators where n' is given by the formula $n' = 1 + (n + 1)d$. Thus the first homology group $V = H_1(k, d)^{ab} \otimes \mathbb{Q}$ has dimension $(n + 1)d + 1$ over $\mathbb{Q}$ and is a module over the group ring $\mathbb{Q}[q]/(q^d - 1)$. If we go modulo the action of the linear transformation $(1 + q + \cdots + q^{d-1})$ the resulting quotient of the homology group of $K(k, d)$ is denoted V^{ni}. Then, as before,

$$V^{ni} = R_d^{n+1},$$

is a free module over the ring R_d.

The natural inclusion of F_{n+1} in the free product $F_{n+1} * t^{\mathbb{Z}}$ induces a map ϕ of the homology groups of the kernels $K_0(k, d)$ and $K(k, d)$, which is also equivariant for the action of the group ring $\mathbb{Z}[\mathbb{Z}/d\mathbb{Z}]$. Thus we get maps of R_d modules $\phi^{ni} : V_0^{ni} \simeq R_d^n \to V^{ni} \simeq R_d^{n+1}$. Since, as $\mathbb{Q}$-vector spaces, the image of $\phi : V_0 \to V$ has codimension $\geq (n+1)d - nd = d$, it follows that (at the level of "non-invariants") the image of V_0^{ni} in V^{ni} has codimension $\geq d-1$.

6.2 Image of the Gassner module

The pure braid group P_{n+1} acts on F_{n+1}, and acts trivially on the Abelian quotient $\mathbb{Z}/d\mathbb{Z}$. We take the trivial action of P_{n+1} on $t^{\mathbb{Z}}$ and get an action of P_{n+1} on $F_{n+1} * t^{\mathbb{Z}}$. Hence P_{n+1} acts on $K_0(k,d)$ and on $K(k,d)$. The map $K_0(k,d) \to K(k,d)$ is equivariant for this action as well, and the action of P_{n+1} on the modules $K_0(k,d)^{ab}$and $K(k,d)^{ab}$ is equivariant for the action of the product group $\mathbb{Z}/d\mathbb{Z} \times P_{n+1}$.

We had the exact sequence

$$1 \to K \to F_{n+1} * F_{n+1}^{ab} \to F_{n+1}^{ab} \to 1,$$

and realised K^{ab} as a module over the group algebra $R = \mathbb{Z}[F_{n+1}^{ab}]$. By replacing R by the larger ring $R' = R[\frac{1}{(1-X_1)\cdots(1-X_{n+1})}]$, we found the direct sum decomposition

$$K^{ab} \otimes_R R' \simeq W \otimes R' \oplus R' v_{n+1},$$

of R' modules (*but not of P_{n+1} modules*). In this decomposition, W was the reduced Gassner representation $g_n(X) : P_{n+1} \to GL_n(R)$. We now get a map from $F_{n+1} * F_{n+1}^{ab}$ onto $F_{n+1} * t^{\mathbb{Z}}$ given by $x_i \mapsto x_i \in F_{n+1}$ and $X_i \mapsto t^{k_i}$. This induces a map from the Abelianised kernels K^{ab} into $K(k,d)^{ab}$, with the image of K^{ab} being precisely the specialisation $X_i \mapsto t^{k_i}$; in other words, the image of K^{ab} is the Gassner representation evaluated at $(t^{k_1}, \ldots, t^{k_{n+1}})$. It follows that if v is the invariant vector in K^{ab} then

$$v = (1 - X_1)\varepsilon_1 + \cdots + (1 - X_1 X_2 \cdots X_n)\varepsilon_n + (1 - X_1 \cdots X_{n+1})v_{n+1}.$$

Then its image in $K(k,d)^{ab}$ is also invariant; moreover, if $t_1 \cdots t_{n+1} = 1$ then the image of v lies in the image of W, namely it is the image of the element $\sum_{i=1}^n (1 - t_1 \cdots t_i)\varepsilon_i$.

The polynomials $1 - q^{k_i}$ and $1 + q + \cdots + q^{d-1} = (1 - q^d)/(1 - q)$ are coprime since k_i and d are coprime. Therefore, $1 - q^{k_i}$ is invertible in the ring $R_d = \mathbb{Q}[q]/(1 + q + \cdots + q^{d-1})$, and therefore, the map $R \mapsto \mathbb{Q}[q]/(q^d - 1)$ induces a ring homomorphism $R' \to R_d$ (recall that $R = \mathbb{Z}[X_1^{\pm 1}, \ldots, X_{n+1}^{\pm 1}]$ and that R' is obtained from R by inverting all the elements $1 - X_i$). Hence $V^{ni} = H_1(K(k,d)^{ab}, \mathbb{Q})/(1 + q + \cdots + q^{d-1})$ is naturally a module over R'. Therefore, V^{ni} is the (full) Gassner representation evaluated at

$(q^{k_1}, \dots, q^{k_{n+1}})$. The module V^{ni} contains image of the sub-module $W \otimes R'$, under the map $R' \mapsto R_d$. However, W is spanned by the "commutator" elements

$$\varepsilon_i = \frac{1}{(1-X_i)(1-X_{i+1})}[x_i, x_{i+1}].$$

Denote by ε_i' the image of ε_i in V_0. Therefore, ε_i' lie in the image of the Abelianised kernel $K_0(k,d)^{ab} \otimes_R R'$ (the commutator $[x_i, x_{i+1}]$ certainly lies in the image of $K_0(k,d)$; but the inverted elements of R' map into elements which have inverses in R_d and hence ε_i' lies in the image tensored with R'). Consequently, the image of $V_0^{ni} = K_0(k,d) \otimes \mathbb{Q}$ has codimension $\leq d-1$:

$$V^{ni}/image\big(W \otimes R'\big) = Rv/\big(1 + q + \cdots + q^{d-1}\big),$$

and the latter space has dimension $\leq d-1$.

The conclusion of the preceding Sect. 6.1 now implies that the module V_0^{ni} maps injectively into V^{ni} and that the image is precisely the image of $W \otimes R'$, the reduced Gassner representation evaluated at $(q^{k_1}, \dots, q^{k_{n+1}})$. We have proved the following

Theorem 24 *The representation of P_{n+1} on the homology group*

$$V_0^{ni} = \big(K_0(d,k)^{ab} \otimes \mathbb{Q}\big)/\big(1 + q + \cdots + q^{d-1}\big),$$

is isomorphic to the reduced Gassner representation specialised at $X_i \mapsto q^{k_i}$, and as a module over R_d it is the free module R_d^n (where R_d is the quotient ring $\mathbb{Q}[q]/(1 + q + \cdots + q^{d-1})$). We have therefore, the decomposition

$$V_0 = \bigoplus_{e|d,\ e\geq 2} g_n(k,e),$$

of the (non-invariant part of the) homology of $K_0(k,d)$ as a sum of the reduced *Gassner representations $g_n(k,e)$.*

Denote by $Q_0(k,d)$ the quotient of the free group $K_0(k,d)$ by the smallest subgroup N normalised by F_{n+1} and containing the "unipotent" elements $x_1^d, x_2^d, \dots, x_{n+1}^d$ and the element $(x_1x_2 \cdots x_{n+1})^{d/r}$, where r is the g.c.d. of the sum $k_1 + k_2 + \cdots + k_{n+1}$ and the number d. (If we view F as the fundamental group of the punctures Riemann surface, then F is a subgroup of $SL_2(\mathbb{R})$ and the loops x_i around the punctures are unipotent elements in $SL_2(\mathbb{R})$; this is the reason we have called the x_i unipotent elements. We do not use the fact that F may be viewed as a subgroup of $SL_2(\mathbb{R})$.) The quotient map $K_0(k,d) \to Q_0(k,d)$ induces a corresponding map on the $\mathbb{Q}$-homology:

$$\phi : V_0 = H_1\big(K_0(k,d), \mathbb{Q}\big) \to X_0 = H_1\big(Q_0(k,d), \mathbb{Q}\big).$$

Since the kernel N to the quotient map is, by assumption, normalised by F_{n+1}, it follows that the foregoing map ϕ on homology is equivariant for the action of $\mathbb{Z}/d\mathbb{Z}$; therefore, ϕ is a map of $\mathbb{Q}[q]/(q^d-1)$ modules. Correspondingly, we get a map

$$\phi : V_0^{ni} \to X_0^{ni} = H_1\big(Q_0(k,d),\mathbb{Q}\big)/\big(1+q+\cdots+q^{d-1}\big).$$

The vector space X_0 does not have any invariants for the group $\mathbb{Z}/d\mathbb{Z}$, because invariants in the quotients X_0 are in the image of invariants in V_0 (G is a finite group and the modules are $\mathbb{Q}$-vector spaces). Secondly, by (11), the only invariants in $V_0 = H_1(K_0(k,d),\mathbb{Q})$ are the span of the "unipotent" classes $[x_1^d],\dots,[x_{n+1}^d]$ which lie in the kernel of the map $K_0(k,d)^{ab} \to Q_0(k,d)^{ab}$. Therefore, $X_0^{ni} = X_0$.

The action of the group P_{n+1} on the free group F_{n+1} is such that each generator x_i of F_{n+1} goes into a conjugate of itself (see Sect. 3.2). Moreover, the product element $x_1\cdots x_{n+1}$ is invariant under all of B_{n+1}. Hence the normal subgroup N is stable under the action of the pure braid group P_{n+1}, and therefore, the homology group $H_1(Q_0(k,d),\mathbb{Q})$ is a P_{n+1} module and the map ϕ is equivariant for the action of P_{n+1} as well. We have the following corollary of Theorem 24.

Corollary 3 *The representation of P_{n+1} on the quotient X_0 is a direct sum*

$$X_0 \simeq \bigoplus_{e|d,\ e\geq 2} \overline{g}_n(k,d),$$

of the quotients $\overline{g}_n(k,d)$ of the reduced Gassner representations by the (possibly one dimensional) space of invariants.

Proof Since $X_0 = X_0^{ni}$ it follows that the elements $x_i^d \in V_0$ map to zero in X_0. Hence the quotient X_0 is V_0 modulo the $\mathbb{Z}[q]$-module generated by $g = (x_1x_2\cdots x_{n+1})^{d/r}$: the other elements $[x_i^d]$ in V_0 map to zero.

We now deal with the element $(x_1x_2\cdots x_{n+1})^{d/r}$. Recall that r is the g.c.d. of the integers d and $\sum_{i=1}^{n+1} k_i$. Then, the element $g_\infty = (x_1x_2\cdots x_{n+1})^{d/r}$ viewed as an element of $F_{n+1}/K_{n+1}(d)^{(1)}$ (written multiplicatively) lies in $K_{n+1}(d)^{ab}$. Put $t_i = q^{k_i}$ and $\pi = t_1t_2\cdots t_{n+1}$. In $K_{n+1}(k,d)^{ab}$ the element g_∞ of the Abelian group (written additively) is of the form

$$w = (x_1x_2\cdots x_{n+1})^{d/r} = v\big(1+\pi+\cdots+\pi^{d/r-1}\big) = \lambda v.$$

In this formula, v is the invariant element in K^{ab} encountered before:

$$v = (1-t_1)\varepsilon_1 + \cdots + (1-t_1t_2\cdots t_{n+1})\varepsilon_n.$$

We need only check that this element $g_\infty = \lambda v$ goes to zero in the e-th component of the decomposition $H_1(K_0(k,d),\mathbb{Q}) \simeq \bigoplus g_n(k,e)$, exactly when $g_n(k,e)$ has an invariant vector.

If $g_n(k,e)$ has an invariant vector, then the element $\pi = t_1 t_2 \cdots t_{n+1}$ of $\mathbb{Z}[\mathbb{Z}/d\mathbb{Z}]$ maps to 1 in the quotient $\mathbb{Z}/e\mathbb{Z}$ of the group $G = \mathbb{Z}/d\mathbb{Z}$. Hence the projection to the e-th factor of w is a non-zero scalar multiple of the projection of v since $\lambda = d/r \neq 0$ at the e-th place.

If $g(k,e)$ does not have an invariant vector, then $\pi = t_1 t_2 \cdots t_{n+1}$ is not 1 in $\mathbb{Z}/e\mathbb{Z}$; hence $\lambda = \frac{1-\pi^e}{1-\pi} = 0$ and our element g_∞ goes to zero. We have therefore verified that corresponding to the decomposition

$$V_0 = H_1\big(K_0(k,d),\mathbb{Q}\big)/\big(1+q+\cdots+q^{d-1}\big) \simeq \bigoplus_{e|d,\ e\geq 2} g_n(k,e)$$

of representations of the pure braid group P_{n+1}, the quotient $X_0 = V_0^{ni}/\mathbb{Q}[G]g_\infty$ has a corresponding decomposition

$$X_0 = H_1\big(Q_0(k,d),\mathbb{Q}\big) \simeq \bigoplus_{e|d} \overline{g_n(k,e)}.$$

as representations of the pure braid group P_{n+1}. □

Corollary 4 *If $n \geq 2d$, then the image of the representation of P_{n+1} on the space W_0 is an arithmetic group.*

Proof For $e \geq 2$ dividing d, denote by G_e the unitary group of the skew Hermitian form $\overline{h}$ on the space $\overline{g}_n(k,e)$. We have seen from the preceding corollary that if Γ is the image of the action of P_{n+1} on X_0 then Γ is contained in the product $\prod_{e|d} G_e(O_e)$. Suppose first that $e \geq 3$. Then by Theorem 16 (since $n \geq 2d \geq 2e$), the image of Γ in $G_e(O_e)$ has finite index. If $e = 2$ then the method of Theorem 16 does not apply. However, for $e = 2$, is a well known theorem of [3] that the image of Γ in $G_e(O_e) = Sp_{2e}(\mathbb{Z})$ has finite index. Therefore, by Lemma 10, Γ has finite index in the product $\prod_{e|d} G_e(O_e)$. □

7 Connection with monodromy

7.1 Some cyclic coverings of $\mathbb{P}^1$

Let $a_1, a_2, \ldots, a_{n+1}$ be distinct complex numbers; write S_a for the complement in $\mathbb{C}$ of these points: $S_a = \mathbb{C} \setminus \{a_1, a_2, \ldots, a_{n+1}\}$. The fundamental group of S_a, once a base point is chosen, may be identified with the free group on F_{n+1} generated by small circles x_i going around the point a_i counterclockwise once (and joined to the preferred base point by an arc which avoids all the points a_j and has zero winding number around all the points a_j). The map $S_a \to \mathbb{C}^*$ defined by

$$x \mapsto (x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_{n+1})^{k_{n+1}} = P_a(x),$$

induces a homomorphism $F_{n+1} \to q^{\mathbb{Z}}$, which sends each x_i to q^{k_i}. Here, q is a small circle around zero in $\mathbb{C}^*$ which runs counterclockwise exactly once.

For future reference, note that the loop around infinity lying in S_a represents (the inverse of) the product element $x_1 x_2 \cdots x_{n+1}$ and that this element is invariant under the action of the braid group B_{n+1} on the free group F_{n+1}.

The affine variety $\mathbb{C}^* = \mathbb{G}_m$ admits a cyclic covering of order d given by $z \mapsto z^d$ from $\mathbb{G}_m$ to $\mathbb{G}_m$. The covering may be realised as the space $\{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* : y^d = x\}$ and the covering map is the first projection. Pulling this covering back to S_a we get a cyclic covering of S_a, realised as the space

$$X_{a,k} = \left\{(x, y) \in S_a \times \mathbb{C}^* : \ \ y^d = (x-a_1)^{k_1}(x-a_2)^{k_2}\cdots(x-a_{n+1})^{k_{n+1}}\right\},$$

with the first projection being the covering map from X_a onto S_a. Therefore, under the identification of the fundamental group of S_a with F_{n+1}, the fundamental group of $X_{a,k}$ is identified with $K_0(d, k)$.

As the collection a varies, we get a collection $\mathcal{P}$ of monic polynomials P_a of degree $n+1$ which have distinct roots a_i occurring with given multiplicities k_i, and if $\mathcal{Q}$ denotes the variety

$$(w, x, P) \in \mathbb{C}^* \times \mathbb{C} \times \mathcal{P} : w = P(x),$$

then the projection on to the third coordinate gives a fibration over $\mathcal{P}$ with fibre at P being S_a (here a is the collection of roots of P). We therefore get a monodromy action of the fundamental group of $\mathcal{P}$ on the fundamental group F_{n+1} of the fibre. We have the following basic theorem of E. Artin.

Theorem 25 (Artin) *The fundamental group of $\mathcal{P}$ is a subgroup of the Braid Group B_{n+1} and contains the pure braid group P_{n+1}. The monodromy action of P_{n+1} on $\pi_1(S_a) \simeq F_{n+1}$ is the usual action of P_{n+1} on F_{n+1} defined in section* (3.2).

Consequently, the monodromy action on the fibre of the fibration

$$\left\{(y, x, a) \in \mathbb{C}^* \times \mathbb{C} \times \mathcal{C} : y^d = \prod (x-a_i)^{k_i}\right\}$$

over $\mathcal{C}$ is the usual action of P_{n+1} on the subgroup $K_{n+1}(d) \simeq \pi_1(X_{a,k})$; therefore, P_{n+1} acts on the first homology $K_{n+1}(k, d)^{ab}$ of $X_{a,k}$, and this gives the monodromy action. Therefore, the first part of Proposition 3 follows from (the Hurewicz Theorem and) Theorem 24.

7.2 The compactification of $X_{a,k}$

Denote by $X^*_{a,k}$ the compactification of the affine $X_{a,k}$; $X_{a,k}$ is a compact Riemann surface with finitely many punctures; hence $X^*_{a,k}$ is a smooth projective curve obtained by filing in these punctures.

Now the covering map $X_{a,k} \to S_a$ is such that these punctures lie over the points a_i or else over the point at infinity of S_a. If a puncture lies over some a_i, then the image of a small loop around the puncture in F_{n+1} is x_i^d (since k_i is coprime to d); if the puncture lies above infinity, then the image of a small loop around the puncture in F_{n+1} is a power of $x_1x_2\cdots x_{n+1}$ the loop around infinity; therefore, such an element is invariant under the action of the braid group.

The mapping of $\pi_1(X_{a,k}) \to \pi_1(X^*_{a,k})$ is such that these loops around the punctures generate the kernel, by the van Kampen theorem; consequently, the elements $x_1^d, \dots, x_{n+1}^d$ map to zero in $H_1(X^*_{a,k}, \mathbb{Z})$ and the element $(x_1x_2\cdots x_{n+1})^{d/r}$ maps to 0. Therefore, the fundamental group of $X^*_{a,k}$ may be identified with the quotient $Q_0(k,d)$ of Corollary 3. Therefore, the second part of Proposition 3 follows from Corollary 3.

The arithmeticity of the monodromy (Theorem 1) now follows from Corollary 4 since the homology of $X^*_{a,k}$ is the homology of $Q_0(k,d)$.

Acknowledgements I am very grateful to Madhav Nori for the crucial remark that an earlier proof for the arithmeticity of the image of the Burau representation would also go through for the Gassner representation).

The support of the J.C. Bose fellowship for the period 2008–2013 is gratefully acknowledged.

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